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ABSTRACT

The third volume of this proceedings contains more of the research reports begun in Volume 2. Papers include: (1) "A Semiotic Model for Analyzing the Relationships between Thought, Language and Context in Mathematics Education" (Juan D. Godino and Angel M. Recio); (2) "Conceptions as Articulated in Different Microworlds Exploring Functions" (Veronica Gitirana Gomes-Ferreira); (3) "The Nature and Role of Discussion in Mathematics: Three Elementary Teachers' Beliefs and Practice" (Susie Groves and Brian Doig); (4) "Proof in Geometry as an Explanatory and Convincing Tool" (Nurit Hadas and Rina Hershkowitz); (5) "The Case of Rita: 'Maybe I Started To Like Math More'" (Markku Hannula); (6) "On Teaching Early Number through Language" (Dave Hewitt and Emma Brown); (7) "Relating Culture and Mathematical Activity: An Analysis of Sociomathematical Norms" (Lynn Liao Hodge and Michelle Stephan); (8) "'Automatism' in Finding a 'Solution' among Junior High School Students" (Bat-Sheva Ilany and Nurit Shmueli); (9) "The Role of Context in Collaboration in Mathematics" (Kathryn C. Irwin); (10) "Design and Evaluation on Teaching Unit: Focusing on the Process of Generalization" (Hideki Iwasaki, Takeshi Yamaguchi, and Kaori Tagashira); (11) "The Teaching-Research Dialectic in a Mathematics Course in Pakistan" (Barbara Jaworski and Elena Nardi); (12) "Characterizing Mathematics Teaching Using the Teaching Triad" (Barbara Jaworski and Despina Potari); (13) "The Mediation of Learning within a Dynamic Geometry Environment" (Keith Jones); (14) "Using Information Systems for Mathematical Problem-Solving: A New Philosophical Perspective" (C. Jotin Khisty and Lena Licon Khisty); (15) "Collaborative versus Individual Problem Solving: Entering Another's Universe of Thought" (Carolyn Kieran and Tommy Dreyfus); (16) "Increasing Teachers' Awareness of Students' Conceptions of Operations with Rational Numbers" (Ronith Klein, Ruthi Barkai, Dina Tirosh, and Pessia Tsamir); (17) "Inflexibility in Teachers' Ratio Conceptions" (Anat Klemer and Irit Peled); (18) "Preservice Teachers' Conceptions of Probability in Relation to Its History" (Hari Prasad Koirala); (19) "Team Teaching and Preservice Teachers'

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Volume 3

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A Semiotic Model for Analysing the Relationships Between Thought, Language and Context in Mathematics Education

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__Angel M. Recio__, University of Córdoba (Spain)

Abstract:

In this research report, we propose a semiotic and epistemological model for studying the processes of interpretation and communication in mathematics education. This model is based on the notion of semiotic function defined on three types of primitive mathematical entities: extensional, notational and intensional entities. We also identify various types of meanings depending on the nature of the semiotic function contents. An example of this model, as well as the multiplicity of codes in mathematical work, is shown by analysing the semiotic processes which take place when solving an elementary geometry task.

1. The relationships between semiotics and epistemology in mathematics education

Growing interest has recently been noted in the use of semiotic notions for studying teaching and learning processes within the mathematics education research community. Examples of this interest are some of the projects presented at the PME Conferences (Ernest, 1993; Vile and Lerman, 1996), and those carried out by Bauersfeld and colleagues (Cobb and Bauersfeld, 1995) from the perspective of symbolic interactionism, which considers the notions of meaning and negotiation of meanings to be central to mathematics education. We also highlight the published research concerning the influence of mathematical language in mathematics teaching and learning (Ellerton and Clarkson, 1996), and the investigations about understanding mathematics (Sierpinska, 1994; Godino, 1996), which cannot avoid the issues of meaning.

We consider, however, that more investigations from the perspective of the mathematics education are needed on the notion of meaning and on the relationships between the semiotic and epistemological components involved in mathematical activity, that is, on the nature and type of objects whose meanings are involved therein. We consider it necessary to carry out a more extensive and in-depth study of the dialectical relationships between thought (mathematical ideas), mathematical language (sign systems) and problem-situations for the solution of which such resources are invented. In fact we would have to progress by developing a specific semiotics to study the interpretation processes of mathematical sign systems involved at the heart of didactic systems. In mathematical work, symbols (signifiants) convey or substitute conceptual entities (meanings). Even when this is also important, the crucial point in mathematics instruction processes is not just mastering the syntax of the mathematical symbolic language, but mastering its semantics. That is to say, understanding the nature of mathematical concepts and propositions and their relationship to contexts and problem situations which emerge from their solutions. Furthermore, it is necessary to elaborate theoretical models to link the semiotic dimension in mathematics education (their syntactic, semantic and pragmatic aspects), to the epistemological, psychological and sociocultural dimensions. These models require taking, amongst others, the following elements and assumptions into account:

- Diversity of objects involved in mathematical activity, both on the plane of expression, and in that of content (conceptual, notational, and situation problems).

- Diversity of acts and processes of semiosis between the different types of objects and ways of producing signs.

- Diversity of contexts and spatial, temporal and psychosocial circumstances that determine and relativise semiotic processes.

In Godino and Batanero (1994) and (1997), we developed a theory regarding the meaning of mathematical objects, from pragmatic and anthropologic assumptions, and considering such objects as signifiants of the "system of practices carried out by people when faced with a certain class of situation-problems". We attributed a diachronic and evolutionary character to this meaning, depending on institutional contexts and personal circumstances. This notion of meaning could be useful for describing certain interpretative processes, particularly in the stages for designing, developing and evaluating teaching and learning plans for mathematical contents. Beyond the interpretation of conceptual entities required in the communicative processes carried out in mathematics education, the expressive means and problematic situations themselves give rise to interpretations by the message receivers at a given time and under certain circumstances. There, meaning has a synchronous and local character: It is the content that the speaker of an expression refers to, or the content that the receiver interprets. In other words, what one means, or what the other understands.

In this research work, we analyse this local use of the term "meaning", starting from a personal interpretation of the epistemological triangles (Steinbring, 1997; Ogden and Richard, 1923) and applying the notion of semiotic function developed by Eco (1979). This will allow us to identify three basic types of elementary meanings taking into account the different natures of semiotic correlation content, which we shall call notational, extensional and intensional meaning, depending on whether the correspondence final object is an external representation (an ostensive representation), a problem situation, or a mathematical abstraction.

2. An interpretation of the epistemological triangle

The relationships between the signs used to codify knowledge and the contexts that are needed to establish their meaning has been modelled by several authors using triangular schemes. Amongst these schemes, we highlight those proposed by Frege, Peirce, Ogden and Richards, as well as their interpretation by Steinbring, which he calls the epistemological triangle. The elements included by Steinbring in this triangle are concept, sign/symbol and object/reference context.

Taking into account this triad, as well as the conceptual triplet by Vergnaud (1982), we shall draw a theoretical model including the following basic entities:

Phenomenologies, that is, problem-situations, applications, tasks; in general, extensional entities inducing mathematical activities.

Notations, that is, every material and ostensive representation used in mathematical activity (terms, expressions, symbols, graphics, tables, graphs, etc., in general, notational entities)

Generalisations, mathematical ideas, abstractions (concepts, propositions, procedures, theories, i. e., intensional entities)

In mathematical work, generalisations and problem situations are given by notations that describe their characteristic properties. Both entities are inseparable from the ostensives, in which they are embodied but with which the are not identifiable, that is, we consider that mathematics may not be reduced to the language with which it is expressed. *"The stated fact must be distinguished from its statement"* (Searle, 1997; p. 21).

The categories of entities proposed are related to the theory of conceptual fields by Vergnaud (1982), who considers a concept as a triplet formed by the "*set of situations that make the concept meaningful, the set of invariants that constitute the concept and the set of symbolic representations used to represent the concept, its properties and the situations it refers to* "(p. 36)

We conceive mathematical generalisations and phenomenologies in the terms used by Freudenthal (1982) the *noumena* and *phainomena*. Mathematical objects are noumena for this author, that is, objects of thought (ideas), such as numbers. Mathematical concepts, in general mathematical structures serve to organise phenomena of concrete and mathematical worlds. We consider that the mathematical study of such phenomena confronts the person with problem situations, from which we establish the connection between both notions.

The notion of mathematical generalisation described by Dörfler (1991) is another starting point for us to interpret mathematical generalisations (noumena) as products of processes for generalising the subject's actions when involved in certain classes of problem situations. They are, therefore, not mere empirical generalisations of features common to objects or situations, but generalisations of schemes or action system invariants, as well as performance conditions and results from such actions, supported by sign systems.

Notational entities can be letter or number chains, graphs, diagrams, or even physical objects. These notational systems frequently play the role of "representation systems", which means, they replace something else or some aspect of another entity. However, in our model, this representation role is not exclusive to this class of objects (mathematical ideas and situations can also be signs of other entities). Notational systems do not only have a semiotic value but they are also ostensive instruments for mathematical activity.

In addition, the semiotic and epistemological model outlined requires another primitive element to describe and explain interpretation and communication processes in mathematics education. This is the notion of *pragmatic context* that we conceive in a very general way as the set of extra and intra-linguistic factors sustaining and determining mathematical activity, and, therefore, the form, suitability and meaning of objects involved therein. It includes the various conditioning aspects of mathematical activity described in a social constructivist account of mathematics (Ernest, 1993).

3. Semiotic functions

In mathematical work we usually take some objects to represent others, especially abstract objects, and a correspondence, frequently implicit, exists between the representative and the objects represented. "*There are words, symbols or other ostensive objects that mean or express something, represent or symbolise something else besides them, and make it publicly understandable*" (Searle, 1997, p. 76).

According to Eco "there is a *semiotic function when an expression and a content are in correlation, and both elements are turned into FUNCTIVES of the correlation*" (Eco, 1979, p. 83). Such a correlation is conventionally established, though this does not imply arbitrariness, but it is coextensive to a cultural link. There may be functives of any nature or size. The original object in the correspondence is the *signifiant* (plane of expression), the image object is the meaning (plane of content), that is, what is represented, what is meant, and what is referred to by the speaker.

Types of semiotic functions

The three types of primary entities considered (extensional, intensional and notational entities) can play both the role of expression or content in semiotic functions. Hence, nine different types of such functions are applicable. Though some of these functions can be clearly interpreted as specific cognitive processes (generalisation, symbolisation, etc.), in this work we will classify and characterise these functions regarding the plane of content (meaning); therefore these nine types are reduced to the three described below.

(1) Notational meaning: Let us call a semiotic function notational when the final object (its content), is a notation, that is, an ostensive instrument. This type of function is the characteristic use of signs to name world objects and states, to indicate real things, to say that there is something and that thing is built in a given manner. The following examples show this type of meaning:

- When a particular collection of five things are represented by the numeral 5.

- The symbol Pn (or n!) represents the product n (n-1) (n-2) ... 1

- In the phrase, "In the histogram of figure x, which is the absolute frequency of the modal interval?" The word /histogram/ refers to another ostensive object that is shown in the figure. Also the expressions /absolute frequency/ and /modal interval/ refer to ostensive observable objects in the figure (a number labelled on the ordinates axis, an identifiable interval on the abscissas axis.

(2) *Extensional meaning*: A semiotic function is extensional when the final object is a situation - problem or a phenomenology, as in the following examples:

- As a rule, the verbal, graphical or mixed description of a situation - problem. Such a description is a different object of the situation itself.

- The simulation of phenomena (i.e., it is possible to represent a variety of probabilistic problems with urn models)

- Using rule and compass constructions (or computer environments, like Cabri) to represent elementary geometry problems.

(3) *Intensional meaning*: A semiotic function is intensional when its content is a generalisation, as in the following examples:

- In the definitions of a concept, for example, "an angle is a pair of rays with the same origin" , the word /angle/ refers to an abstract object.

- In expressions such as, "Let μ be the mathematical expectation of a random variable ξ", or "Let $f(x)$ be a continuous function ".

The notations μ, ξ, $f(x)$, or the expressions /mathematical expectation/, /random variable/ and /continuous function/, refer to mathematical generalisations.

- "The histogram is used to represent a frequency distribution of a statistical variable grouped into intervals of class ". Here the word /histogram/ refers to a generality.

Every intensional and extensional semiotic function can be interpreted, furthermore, as a notational correspondence and vice-versa, since abstractions and the situations - problems are textually fixed.

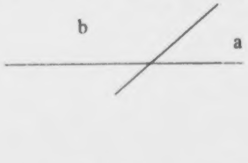
4. Applying the theoretical model to the analysis of a geometry task

In this section we apply the theoretical model described for analysing a geometry proposition statement and the process followed by a student to solve it. This task was included in a written test to assess university students' generalisation and reasoning capacities (Recio and Godino, 1996). Below we reproduce the task:

"Prove that the bisectrices of two adjacent angles form a right angle. (Remember that two adjacent angles have a common vertex and side, and they add up to 180°. A right angle measures 90°. The bisectrix of an angle is the ray that splits it into two equal parts)".

The response given by a student was the following:

If a and b are the two adjacent angles, then a+b = 180°. Consequently, the angle that forms the bisectrices is:



$$\frac{a}{2}+\frac{b}{2}=\frac{a+b}{2}=\frac{180^\circ}{2}=90^\circ$$

Objects involved

The statement of the task denotes a general property of adjacent angles (a generalisation) and also describes a problem situation for the subjects: building an argument to support the universal and intemporal need for the truth expressed in the statement.

From a pragmatic viewpoint, the context in which the task is proposed by the researcher (transmitter) also produces interpretative processes on the part of the students (receivers of the same). In this work we shall focus, however, on identifying the objects and interpretations required by the different terms and expressions used in the statement and the solution. The words and expressions raising interpretative processes are the following:

Prove, angle, vertex of an angle, sides of an angle, ray, bisectrix of an angle, angle equality, right angle, straight angle, angle sum, adjacent angles, division of an angle into parts, measure of an angle, the degree as a unit of angle measure, 180°, 90°.

These terms and expressions denote conceptual entities or mathematical operations controlled by definitions that the subject competent in mathematics should recall (implicitly in general) and so apply them in the task and circumstances requested.

Interpretative processes:

From a didactic-mathematics point of view we can identify the interpretative processes described below. This relation of semiosis acts constitutes a pattern for assessing the processes that students can or cannot carry out in each particular case. Their observation would require performing clinical interviews with the subjects.

I1: To prove means to establish the universal and intemporal truth of the theorem (the angle that forms the bisectrices is right for any two adjacent angles)

I2: The segments of the figure represent two adjacent angles (It is implicitly assumed that the segments represent the sides (rays) of the angles).

I3: The letters a, b designate variables, and since they are written near the angles in the Figure, they represent any pair of adjacent angles.

I4: ° represents an angle of a particular size taken as a unit of measurement.

I5: 180° is the result of adding 180 angles with an amplitude of one degree.

(It is the measure of the angle known as straight angle taking the degree as a unit).

I6: $a+b$ represent the sum of the two adjacent angles.

I7: $a+b=180^\circ$; translate the expression, "the sum of any two adjacent angles".

I8: The symbols $a/2$ and $b/2$ represent any of the two angle regions in which the bisectrices of the adjacent angles divide into a and b, respectively.

I9: $a/2+b/2$ represent the sum of the two angles.

I10: $(a/2)+(b/2) = (a+b)/2$; adding two halves of two angles produces the same angle as finding half the angle formed by adding the two angles given.

This is one possible interpretation of the equality. But it can also be interpreted that the equality is due to the distributive property of the product of numerical scalars with respect to the sum of the angles (Structure of semimodule of the angle size magnitude).

I11: $(a+b)/2 = 180^\circ/2$; notational transformation (deictic), taking into account the interpretation I6.

I12: $180^\circ/2=90^\circ$; it means finding half of the angle for which the measurement in degrees is 180. An angle for which the measurement is half of 180, or rather 90 is obtained.

I13: 90° is the measure of the right angle; it is the result of adding 90 single degrees together; this result is the right angle (half of the straight angle).

In this process we have changed to the numerical-algebraic register by introducing the angle measurements. This change serves to associate rational numbers to angle quantities (180 to straight angles, 90 to right angles, a/2 and b/2 to the measures of the adjacent angles) and hence to operate in this register. Finally, the number obtained, 90, must be interpreted as the measurement of the right angle, and this, as being half the straight angle.

In each act of semiosis we can recognise the dialectic between ideas, language and phenomenological context. For example, in I3, the idea of variable is "incapsulated" in the notation 'a', 'b', and "exemplified" by a pair of adjacent angles, which constitute in this act their phenomenology. In I1, the idea of proof (to establish the universal and timeless truth of a proposition) is incapsulated in the expression 'to prove' and exemplified with the bisectrices of the adjacent angles always forming a right angle, which constitutes a phenomenology associated with the idea of proof.

5. Conclusions

The semiotic-epistemological model described in this research report - from the didactic of mathematics perspective - is based on the notion of semiotic function as suggested by Eco and by a personal interpretation of the epistemological triangles described in the bibliography. It has allowed us to outline a theory of the meanings of objects involved in mathematical activity. Its application to the analysis of a didactic episode has shown the ontological and semiotic complexity of mathematical activity, even in the elementary task chosen. We think that this kind of analysis, by showing the multiplicity of codes involved, can help to overcome a certain "illusion of transparency" regarding abstraction and reasoning processes in mathematics teaching and learning. This will let us identify critical points and conditioning factors of semiotic acts and confront them using suitable didactic actions.

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Conceptions as articulated in different microworlds exploring functions

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Abstract

This paper compares the ways students interact with three microworlds exploring real functions of real variables, which were designed around two software programs which explore the dynamic potential available through computer environments. It reports results of case studies of the development of pairs of students while interacting with these microworlds. The way the students' conceptions are articulated when situated in the different microworlds showed to be influenced by pedagogical and technical aspects of the microworlds. The role of articulating these situated conceptions is, also, examined by the gains for students' development while connecting these conceptions with their previous knowledge.

Introduction

Microworlds, technically described as a computer environment which embodies a domain of knowledge, have been explored in mathematics education to provide access to ideas and phenomena that are not easily find in other media by students. A microworld represents mathematical concepts in a peculiar way that can be close or far away from the school mathematics. Hoyles & Noss (1993) had observed that "students frequently construct and articulate mathematical relationships which are general within the microworld yet are interpretable and meaningful only by reference to the specific (computational) setting" (pp.84). Thus, it is important to discuss the role of allowing students to explore a microworld if the understanding built within it lacks universality.

This paper will discuss results of case studies (Gomes Ferreira, 1997)[1] which compare different "ways in which learners structure their own learning, as well as on the ways in which the setting structures it" (Noss & Hoyles, 1996: 108) while exploring different microworlds on the concept of real function with real variable. It also focuses on how connections were forged.

Brief description of the research

This case studies research investigated how students articulated knowledge of mathematical function while interacting with microworlds, created from a set of activities developed to encourage exploration of the dynamic features of two software programs and discussion within pairs of students. Instead of the concept, a selection of

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[1] This research was supported by CAPES (a Brazilian Research Foundation) as a PhD research

properties of function (range, periodicity, variation, turning point and symmetry) was distinguished.

Two software programs which exploit the possibilities of computers to explore representations of functions by continuous movement were selected: DynaGraph (DG) (Goldenberg et al, 1992) and Function Probe (FP) (Confrey et al, 1991). Function Probe is a multiple representational software, developed by the Mathematics Education Research Group at Cornell University, which allows continuous and direct transformations of graphs. DynaGraph is a visual representation of function software, developed by the 'Educational Development Center', which allows students to vary the variable of a function and observe the variation of its output.

Four pairs of Brazilian students, who had already studied functions at school, followed a sequence of activities during 13 meeting with 2 hours each: A pre-test, focusing on the chosen properties; familiarization with the research environment; the microworld activities in two different sequences (two pairs did the activities in both DG Parallel and DG Cartesian followed by the activities in FP, and the other two pairs followed the activities in the opposite order); a final interview to investigate how students made links during the activities as well as to motivate new links if possible. The set of activities will be detailed after the description of the programs.

Starting with an examination of the curriculum followed by the students as a means to describe the origins of their conceptions, a longitudinal investigation was undertaken in order to identify the main features of each of the microworlds that appeared to contribute to the students' progress. The students' conceptions were analysed by drawing attention to their origins, their usefulness and their potential limitations (from a mathematical point of view). A methodology for this longitudinal study was devised which incorporated visual presentations to capture the main characteristics of students' conceptions (see Gomes Ferreira & Hoyles, 1997).

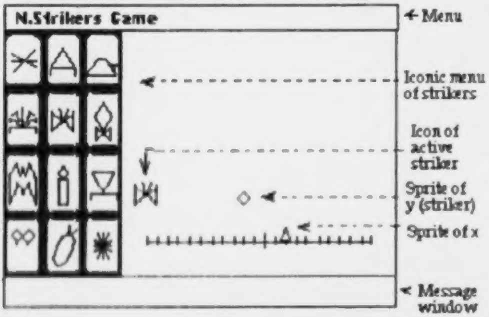
On discussing the role played by a computer environment in student's learning, Noss & Hoyles (1996) have been working on the web and situated abstration framework - a framework to understand how meanings are constructed by actions on virtual objects and relationships while students explore computer environments. One develops an overview of the Web of mathematical ideas by 'navigating' on a computer environment and forging familiar local connections, thus, developing an individual understanding.

This framework recognises the contingency of the domain. Students "construct mathematical ideas by drawing on the webbing of a particular setting which, in turn, shapes the way the ideas are expressed" (Ibid.: 122). On attempting to describe these constructions Hoyles & Noss (1987) coined the term situated abstraction. Situated abstraction refers to the object as well as the process articulated by a student in a domain. So, the question discussed here is how students structure their understanding while working within DG Parallel, DG Cartesian and FP microworlds as well as how these microworlds structure students' learning.

DG Parallel and DG Cartesian programs

As the parallel version of DynaGraph, DG Parallel represents a function point-by-point by two sprites. One of them corresponds to the input of the function (in general denoted by x) and the second sprite represents the image of the function (f(x) or y).

Figure 1 — Screen of DG Parallel with the striker of $f(x)=-x$ displayed



Differently from DynaGraph, DG Parallel explores twelve functions hidden in behaviour of strikers which are presented by different icons. Using the mouse, the student moves (varies) x horizontally, obtaining as feedback the corresponding movement (variarion) of y according to the chosen function which is represented by the active striker. Thus, the software can facilitate students to develop a co-variational view of functions. By clicking the mouse on the iconic menu, users can change the active striker (active function). By displaying the icon of the active striker, DG Parallel enables students to remember which 'function' is on. More than one striker can be choose to compare the behaviour of different strikers.

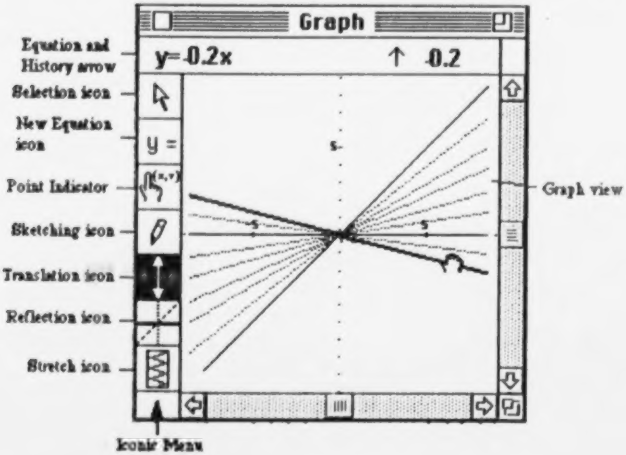
DG Cartesian is similar to DG Parallel. The same functions are hidden in the same strikers. It mainly differs from DG Parallel in that: the axes appear as in the Cartesian system and a dot representing (x,y) is added. In both DG Parallel and DG Cartesian, students have no access to any other representation of them, in particular any algebraic representation.

Function Probe

Function Probe is a multiple representational software tool to enable students to explore real functions. It combines three representations (algebraic, graphic and tables) in three windows (Graph, Table and Calculator). This study focused on the Graph window of FP particularly in the transformations students do in graphs while looking for properties of functions. For a complete description of FP see Confrey et al (1991).

The Graph window presents a iconic *menu of commands*, which allows actions within graphs. Among these actions, there are the dynamic transformations within graphs: vertical and horizontal stretches; vertical and horizontal translations; and vertical, diagonal and horizontal reflections. Figure 2 tries to reproduce, in a static way, a dynamic vertical stretch from the graph of y=x to the graph of y=-0.2x.

Figure 2 - *Vertical stretch* from the graph of y=x to the graph of y=-0.2x



The Activities and the Final Interview

The design of each microworld was a result of conceptual, technical, cognitive, and pedagogical considerations. In each microworld, the activities had similar structure divided in three phases. The first phase was a session to familiarise the students with the software commands. In the second phase, the activities used were descriptions to lead students to explore the properties of twelve functions chosen into four families (constant; linear; quadratic and sine functions) according to their representations: as graph in FP and as behaviour of strikers in both DG microworlds. The description activity was made in a describing/guessing form, in which one student was expected to guess what was the function chosen and described by his/her partner. In the last phase, the students were required to group the functions as represented in each of the microworlds according to the properties they had observed.

The choice of the sample of functions to be used also played an essential role in the construction of the microworlds. The students had already studied each of the chosen families of functions. In order to choose the functions two criteria were considered: the properties had to be emphasised by the sample; the dynamic potential of FP, DG Parallel and DG Cartesian could be used when exploring the functions in the activities.

In the final interview, the students were asked to: match the strikers with the graphs; identify conceptions built within DG microworlds in the graphs; predict the behaviour of a new striker which corresponded to a graph transformed from another using FP, having the behaviour of the striker corresponding to the graph. All the activities are derived from suggestions presented by Kaput (1992) as ways to motivate connections.

Conceptions Articulated in DG Microworlds

Common conceptions among the pairs of students made clear the 'special status' attributed to some properties by 'the motions of x and y' in DG Parallel indicating an acquisition of a co-variational conception of function (Goldenberg et al, 1992), although this depended on the sub-concept in question. For example, line symmetry and periodicity were rarely identifying in DG Parallel. A summary of the conceptions developed by the majority of the pairs in DG Parallel shows the special status of the motions of x and y in DG Microworlds: turning points started to be identified as '*point where y changes orientation*'; horizontal straight lines were justified by 'y is independent of x' and '*y is motionless*'; monotonicity as 'direction of straight lines' was interpreted by '*comparing orientations of the motions of x and y*', what facilitated its generalisation to non-linear functions; slope of linear graphs was discriminated by '*comparing the ratio between the variations (or values) of x and y*', linking it to inclination; curved and straight graphs were characterised and justified by constant and variable '*ratio ...*'; Range changed from a polarised approach (positive and negative range) to an approach involving 'bounded or boundless range'.

Bernard & Charles conceptions of monotonicity will be discussed, here, to illustrate the ways the students articulate knowledge in DG Microworlds.

Bernard & Charles: monotonicity

Bernard & Charles' development in monotonicity were quite close with the other pairs. While exploring all the microworlds and in the pre-test, Bernard & Charles restricted the use of the terms 'increasing' and 'decreasing' for linear functions.

However, Bernard & Charles also identified monotonicity in a co-variational way in their pre-test and in DG Parallel, which remained isolated from the others till the final interview. These conceptions seem to have been triggered by their previous understanding of the term 'increasing'. In DG Parallel, Bernard & Charles discriminated monotonicity by comparing the 'orientations of the motions of y and x' (y follows x and y doesn't follow x). The fact of trying to control the orientation of the motions of y while moving x led the students to distinguish these ideas as an invariant to characterise the strikers (functions), even before knowing that each striker hides a function.

From a mathematical viewpoint this co-variational view had the advantage to be generalised among other families of function such as parabolas. The comparison of these orientations of motions was used by the students to distinguish strikers of linear functions from quadratic and sine functions.

Bernard & Charles' great discovery happened in the final interview. They brought the generalisation developed in DG Parallel to the Cartesian system, but it was not straightforward. Firstly, they linked the term increasing with '*direction of the graph*' to '*y follows x*' for increasing and '*y does not follow x*' for decreasing limited to linear functions. As they had this conception ('*y follows x*' or '*y does not follow x*') for strikers given by parabolas, they brought back to graphs the link, using the terms 'increasing' and 'decreasing'. This allowed Bernard & Charles to overcome the obstacles created by using the terms 'increasing' and 'decreasing'.

DG Parallel, a new representation

Given that few attempts were made to build connections to 'old' knowledge during interactions with DG Parallel, this microworld could be explored as a 'new' representation where students appeared more free of previous conceptions. This allowed them to revise conceptions and to generalise some conceptions to a wider set of functions within DG microworlds. In the case when later connections with previous knowledge were built, such as the case of Bernard & Charles investigating monotonicity, the developed conceptions proved to be robust enough to allow students to contrast them with those derived from school knowledge. Thus, a key to the use of qualitatively different multiple representations is synthesis but also articulation of situated abstrations.

Conceptions articulated within FP

The interaction with transformations of graphs in FP led the students to a different way of articulating their conceptions of the properties. In contrast to DG microworlds, the point of exploring the transformations of graphs in FP was to give the students tools to explore, not to shape conceptions. One example will be detailed here.

Bernard & Charles: monotonicity

Bernard & Charles started their explorations of FP classifying 'directions of a straight line' into two types: increasing direction and decreasing direction (without distinguishing the different slopes). This division can be considered as a compartmentalisation in their understanding when the slope was not considered.

Interactions with dynamic transformations of graphs in FP prompted Bernard & Charles to realise the connection between monotonicity and derivative. While investigating the idea of increasing, a horizontal stretch in the graph of y=x (as shown in figure 2) encouraged the students to connect 'direction of straight lines' and 'slopes'. For instance, Charles argued that the change from increasing to decreasing depends on where you have the graph (i.e., where you choose to finish the transformation). He explained that anyway the command changes the direction of the graph but it can pass from one type to the other type.

FP: Tools to explore own conceptions

In FP, the students' conceptions of the properties could not really be categorised in relation to each command explored. The fact that the students were discussing while transforming graphs more often determined the changes in their conceptions, than the command per se. What was revealed in this research were patterns emerging from the ways the students used the commands to modify their own conceptions.

The students used the transformations as tools: to generate and check their own hypotheses by generating examples and counter-examples; to recognise and revise differences in conceptions previously associated, to discover new aspects of a property; to realise limitations of their own conceptions; to generalise their conceptions among different functions; to develop 'comparative measures' for properties they previously perceived pictorially; and to realise relationships between different properties which had previously been compartmentalised. Note that the possibility of generating new examples in FP make its explorations qualitatively different than the exploration of DG microworlds.

Patterns in ways of synthesising

The research came up with some patterns in the ways which led the students to synthesise. In the case of DG Cartesian, the students made the connections by matching the strikers with the family of functions and bringing terms explored at school in these families to the discussion. This last was also exhibited in FP. The fact of working with the same sample of functions in different microworlds also encouraged the students to make connections.

In the case of FP, the students were more open to making connections in response to: the analysis of variants and invariants and the observations of algebraic and Cartesian representation while transforming graphs, which then showed the great importance of the dynamic transformations of graphs for the students in building the connections; the attempts to make sense of results obtained from transformations which were counter-examples of their own assumptions, which then demonstrated that students were stimulated into making connections when their expectations were contradicted; the comparison of two or more functions, which then highlighted the importance of the activities of describing and classifying functions in leading them to connect conceptions.

The two activities of the final interview provoked the students to make their own connections by: linking conceptions which before had been isolated within different microworlds, generalising conceptions previously restricted to one family of functions and revising naive links. The activity of predicting a striker corresponding to a transformed graph also led the students into a new search for conceptions in DG Parallel which they brought to the research environment by Cartesian representation.

Conclusion

The results of students' development depended not only on the computer features but also on the students' interactions during the activities. One illustration of this can be given by the fact that the development of ways to measure, such as the 'ratio between the variations of x and y'. The students had to be precise in comparing two or more functions in order to allow their partners to guess the function described.

The results showed that DG Parallel, a 'new' representation, prompted the development of conceptions free of previous limitations and sufficiently robust to allow revision. However, properties previously perceived pictorially were rarely identified in DG Parallel. Together with DG Cartesian, interactions with this microworld provoked the students to develop a co-variational view of some of the function properties. By way of contrast, using the tools in FP to transform graphs seemed not to shape conceptions, but to assist in the exploration of the function properties.

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THE NATURE AND ROLE OF DISCUSSION IN MATHEMATICS: THREE ELEMENTARY TEACHERS' BELIEFS AND PRACTICE

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There have been many calls for the improvement of discussion or dialogue in mathematics classes. However there has been difficulty in transferring practices developed in research to the wider educational community. This paper reports a small study that assumes that knowledge of current practice is a necessary first step in producing such transfer. Three teachers were video-taped while conducting mathematics lessons and subsequently interviewed in an attempt to establish their current beliefs and practices with respect to the nature and role of discussion. The analysis reveals a concurrence of teachers' intentions and goals, together with a diversity of strategies for achieving them.

Introduction

Greeno (1992) argues that the task of school learning in mathematics and science should be to enhance children's thinking and that, in order to achieve this, classroom activities should be organised as mathematical or scientific discourses. According to Bereiter (1994), classroom discourse can be progressive in the same sense as science, with the generation of new understandings requiring a commitment from the participants to working towards a common understanding which is based on a growing collection of propositions which can or have been tested.

In a similar vein, Cobb, Wood and Yackel (1991) contrast traditional discussion in mathematics classrooms, where the teacher decides what is sense and what is nonsense, with genuine dialogue, where participants assume that what the other says makes sense, but expect results to be supported by explanation and justification.

There have been a number of research and development projects which have attempted to engage students in such dialogue (see, for example, Ball, 1993; Brown & Renshaw, 1995; Cobb, Wood & Yackel, 1991; Lampert, 1990). However, when teachers in the wider education community attempt to implement these ideas in their classrooms, they often only adopt superficial features (see, for example, Stigler, 1996; Knuth, 1994).

In our own work in the *Practical Mechanics in Primary Mathematics* project (see, for example, Doig, 1997; Groves, 1997), we found that many teachers, regardless of their stated theoretical frameworks for learning and teaching, were uncomfortable with whole class discussions based on the notion of scientific dialogue. They appeared to identify any form of whole class interaction with a traditional, expository model of classroom practice, which they claimed to have rejected in favour of small group work with discussion occurring almost exclusively within these small groups.

Heeding the words of Cooney & Shealy (1997, p. 106) that "consideration of what teachers believe and how their beliefs are structured provides us a means of conceptualising teacher education in ways that promote change in something other than a random manner" a large-scale investigation has been proposed, with one of its foci on these beliefs and structures. As a preliminary to this investigation, of the extent to which generalist elementary teachers can develop and implement conceptually coherent personal models of mathematics classroom practice based on the notion of a community of inquiry, we carried out a small-scale study. This study sought to establish what are teachers' beliefs and practice about the nature and role of current classroom discourse in mathematics, science and language.

This paper presents the results of this study through an analysis of teachers' beliefs and practice with respect to the nature and role of discussion as observed in three mathematics lessons and as reported by the teachers in a subsequent interview.

Background and methodology

In order to establish what is current practice in elementary classrooms in Australia, and to what extent this represents an operationalisation of teachers' beliefs, video-data and observations were compared to teachers' responses to face-to-face interviews. The three teachers who took part in this small scale study, were chosen in an attempt to maximise the likelihood of observing practice with features in common with a community of inquiry approach to encouraging dialogue.

The school, at which the three teachers work, is located in a middle-class suburb of Melbourne, and has an excellent reputation for academic results in the local community based on the high quality of its teachers. Two of the teachers, referred to here as Barbara and Anita, were known to one of the researchers for a number of years, and both of these teachers had strongly claimed the use of an open-ended teaching approach, together with the use of sharing and discussion. Barbara had a combined Kindergarten and grade 1 class, while Anita's class was a grade 4 class.

The third teacher, referred to here as Helen, had previously agreed to be observed and interviewed by one of the researcher's post-graduate students and was also known to that researcher through other professional activities. Helen was teaching a grade 5 class.

Each of the teachers had been teaching for 20 to 25 years and were generalist teachers who had taught at most grade levels, although Barbara and Helen had mostly taught at the infant and grade 5 - 6 levels respectively.

One lesson of approximately one hour's duration, in each of the areas of mathematics, science and language was video-taped, and extensive field notes taken during each of the nine lessons. The teachers provided an outline of the aims for each lesson, as well as copies of any worksheets used by the children.

Each teacher took part in a semi-structured interview of between 45 and 60 minutes to determine their beliefs about the learning and teaching of each of these areas, their perceptions of their practice, and their beliefs of the role and nature of discussion. Interviews were audio-taped and later transcribed.

For the purpose of this paper, the data will be taken to consist of the video material and field notes relating to the mathematics lessons, interview data which refer to mathematics, to learning and teaching in general, or which contrasts another learning area with mathematics.

Each video tape was viewed a number of times and partially transcribed during this process. Interspersed with the viewing of the video tape, the interview transcripts were summarised in three stages. Although this is a much smaller study than that reported by Cobb (1995), the analysis of this data also "involved a continual movement between particular episodes and potentially general conjectures" (p. 35), albeit in this case the conjectures usually referred to the connections between beliefs and practice for the three teachers involved. We, too, recognised Lampert's (1990) characterisation of the "zig-zag" path of mathematical activity from conjectures to proofs through the use of counter-examples (Cobb, 1995, p. 35) as applying to our data, when early conjectures about the nature of discussion were refuted by later data.

Data and Results

Results from the analysis are presented here in the form of necessarily brief case studies for each of the teachers. In order to present the results meaningfully the data from all sources for each teacher have been combined to give an overview of each teacher's beliefs and practice.

Helen — grade 5

Helen describes her mathematics teaching as "reasonably traditional", although it has "freed up a lot over the last few years". She sees herself as having a "huge responsibility" in maintaining children's self-esteem and wants "to keep them interested, to get them excited about the topic, to give them the basic nuts and bolts stuff".

A typical lesson starts with "something that is a fun thing" followed after 10 or 15 minutes by "some sort of demonstration on the board". After this, children typically work in groups while Helen works with a small group on the floor. Sometimes there is some "peer tutoring". Helen still wants the grade 5's to use materials and likes "the children to get together and discuss answers". She rounds the lesson off "with us all marking together and discussing that". She believes that children learn maths best by "sharing ideas, doing examples, [and] applying it to another level".

The lesson observed begins with an 11 minute mental computation competition involving pairs of children answering questions such as "three quarters of 32 and six". Children are occasionally asked to explain their reasoning. For example:

Teacher: Point seven of 90 take away six?

Stephen: 57

Teacher: … are you instantly seeing the connections between the numbers? What is the connection?

Stephen: Seven tenths of 90 … I'd say, um, ten tenths is one. It's just nine because all the tenths add up to nine. Then I do 7 times 9 is 63 and I took the six away.

Teacher: You're seeing 7 nines are 63?

Stephen: I just take off the zero.

Teacher: (to whole class) Is point seven less than, equal to or greater than a whole number?

Chorus: Less.

Teacher: So … in that part of the sum it's always going to be less than you started with, less than 90 …

This interaction involves both univocal (transmission) aspects and dialogic (thought-generating) aspects (see Wertsch & Toma, 1995, p. 167). However, the vast majority of the interactions observed in this lesson were of a univocal nature, usually in the "Initiation-Reply-Evaluation" (IRE) form (Mehan, 1979). This was particularly true in the 18 minute "demonstration" segment of the lesson which followed the quiz, with questions such as "If one person is 12° [on the pie chart], how many degrees would five people be?" asked frequently.

Throughout, Helen assumes responsibility for the children's learning. This is particularly evident in the 22 minute segment where children work on the task of producing a pie chart and where most of her time is spent answering children's questions when they have difficulties with the task. She often answers children's questions with questions of her own, but these almost invariably form a carefully selected sequence of IRE interactions. IRE was also the typical interaction pattern in the six minute "sharing time" at the end of the lesson where children displaying their work to the class, were asked closed questions like "What is the biggest section of your pie chart?", "How many hours?", "How many degrees?" , rather than being asked to describe or explain their pie charts in a more conceptual manner.

Helen herself sees the purpose of discussion as "at the beginning … to set the parameters, set the goals, set the guidelines", while "discussion at the end is very important because it ties it all together … who got it, who thinks that they are on the right track, who thinks they need some more time on this…". However sharing time is primarily about "maintaining self esteem". She also sees herself, across all curriculum areas, as a "facilitator … encouraging the children to share ideas among themselves … without dominating", although "a lot of the English discussion comes out of their interests, whereas something like maths or science … you tend to try and keep it on course".

Anita — grade 4

Anita shares the class with another teacher and is responsible for teaching the measurement, chance and data, and space topics only. She says that previously she has "worked a lot with groups", where she would work intensively with one group while the others worked independently. With the current class, because of the "enormous range of ability levels", she has "done quite a bit more class work, where it is an individual challenge. So the kids start off at a fairly low level and some of them will zoom ahead". Anita often has two activities, one of which she identifies as the "focus activity", with half the class doing each so that she can have "just 15 children that I could get around to". She likes the children to use materials and work on open-ended tasks, that sometimes are altered by the children as they work.

The lesson observed begins with an 8 minute introduction to the two tasks, during which Anita says: "Don't forget to keep talking to each other and build on each other's ideas". Children in each half of the class loosely work in groups; for example, the focus activity asks children to make towers with a given number of plastic bricks in two colours and find a way to label each tower to show how many bricks are in their "special" colour. Children in any one group use the same "special" colour and the same number of bricks in each tower. Anita expects that some children will rename and recognise equivalent fractions, while others will increase their understanding of what fractions like 3/10 mean. She tells children that "I am going to be really interested in talking to you as you are working".

Although there is a great deal of talk during the main part of the lesson whilst children are working, there is little evidence that the children work together. In this part of the lesson, Anita circulates and speaks with individual children, frequently asking them how they are going to label their towers and telling them to think about it. The fact that the focus group is involved in such a self-sustaining activity allows her to do this.

On several occasions, Anita stops the class to discuss the task. For example:

Teacher: I want you to look at the labels you've got and see whether or not there might be another way of labelling some of your towers. Ingrid has two labels for one of her towers. What do they say? … Hold up your tower … How much is green?

Ingrid: One half.

Teacher: What's the other label you have for that tower?

Ingrid: Um …

Teacher: You've got it written down there.

Ingrid: Eight over sixteen.

Teacher: Eight ***sixteenths***. Well done. That's to give you a bit of a clue. Can you make some different labels that will still be true about your towers? Stop making towers. Just concentrate on your labels.

Sixteen minutes after this she asks the whole class to listen to a discussion where children with 3 or 4 labels for the same tower discuss their labels. Michael has 1/2, 6/12, 2/3 and 2/4 for the same tower. Anita tells the class that Michael is not sure of one of them and that he is going to check it, which he does some time later. In essence, there is considerable discussion which tries to build on student ideas in this part of the lesson, however it is not clear at times which children are intended to participate.

In the sharing time Anita tries to build up children's self-esteem "by asking something that I know they can contribute" although she acknowledges that "sometimes kids get left behind [in discussion] because they are not following what the others are talking about …". She expects, too, that the children doing the other activity also will learn from the sharing time 'because the second group learns from what the first group has done".

Barbara — Kindergarten & grade 1

Barbara believes that children learn best by doing, having success, using materials and playing or experimenting. She describes a typical mathematics lesson as starting with some oral activities, within which "I might alter the questions I ask children according to their ability". She would then "set the scene" for groups to either work on different activities at their own level, or the whole class on an open-ended task. This is followed by a "sharing time". Both the lesson observed and the one described in answer to the question about a typical activity are strongly linked to content throughout the lessons.

Barbara sees different possible purposes for sharing time. One purpose is exposing children to different ways of finding a solutions (to which she refers frequently in the interview) allowing children to hear others describe tasks they have not yet tried themselves, and a second purpose is that of building self-esteem.

Even though the children are very young (5- to 7-year-olds) Barbara "would be going around saying to them 'how did you find that out, what did you do?' … putting it back onto them to try to explain to me what they have done, getting them to try and help each other". She wants them to feel "that they can say what they think, hopefully with something to back it up, and if they haven't got anything to back it up with, well, maybe they're not right, but it's still fine to give your opinion, and then go back and check". She believes that discussion is important "in helping them form their opinions and be able to justify their opinions" in all curriculum areas.

In the lesson observed, it is clear that both the teacher and the children share these expectations. For example, near the beginning of the lesson, Barbara asks a group of seven children to stand in a line. She then asks Yvonne to find the middle of the line. Yvonne points to Robert, who is in the middle.

Teacher: Is that the middle? How can you prove that? How do you know that's the middle?
Yvonne: Because there's three on that side and three on that side.
Teacher: Does there always have to be three on that side and three on that side?
Yvonne: No.
Teacher: Well how do you know then? What's another way of explaining it? Can you explain it a different way?

Although several children have their hands up at this stage to explain it in a different way, it is not until the discussion has continued for a several minutes, with many different children involved and extra children have been progressively added to each side of the line that Barbara asks the question again:

Teacher: Who can explain that?
Kathy: They're equal.
Teacher: What's equal?
Kathy: The five on each side.

There are many similar occurrences throughout all stages of the observed lesson, while in the sharing time at the end there is an example of children building on one another's solution strategies.

In this classroom, not only are there well-established social norms relating to discussion, but the teacher and children have, in Yackel and Cobb's (1995) sense, socio-mathematical norms for what counts as acceptable explanations and justifications.

Conclusion

Superficially, all three teachers in this study showed remarkable consistency between their beliefs and practice. However, when you view the teachers together, it becomes clear that there may be ways in which they could redefine their practice to be more conducive to achieving their ultimate goals and that some of their current practice might in fact be antithetical to these goals.

For example, while all three teachers saw enhancing children's self-esteem as an important goal, they used substantially different strategies to achieve it. Helen saw sharing of ideas as important for self-esteem, but in fact had few strategies for enabling children to genuinely contribute their own ideas or findings to the discussion. Anita also hoped that children would build on one another's ideas through the sharing time, and allowed 10 minutes for 30 children to individually participate. She lamented that this did not leave her time to follow up children's thoughts and ideas. Barbara, on the other hand, accepts the fact that discussion may expose children "to things that they are not ready for but they can hear a little bit about", but alters her questions to allow everyone to contribute.

It is clear from the diversity of practice of this sample of three teachers, that Cooney & Shealy (1997) are correct in claiming that knowledge of current practice and belief structures is a necessary first step towards effective teacher development. Further, if Yackel's claim (1994, p. 386), that a first priority is to make problematic aspects of teachers' current practice, is correct then knowing this current practice is also essential. We propose to investigate more widely the current beliefs and practice concerning the nature and role of discussion and then, armed with this knowledge, to investigate whether teacher practice can be made problematic through teachers working as a community of inquiry to examine their own and one another's practice.

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PROOF IN GEOMETRY AS AN EXPLANATORY AND CONVINCING TOOL.

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The goal of this paper is to show some mutual relationships between design of activities aimed to put students in geometrical situations where they feel the need for proof, and cognitive research on students' actions in such activities. We exhibit an example of the development process of such an activity in several cycles of design, experimentation and analysis, by describing the work of two pairs of students in two different cycles of the development. We also analyze the task as well as the work of the students. Finally, we discuss principles of activity design for leading students to feel the need for and to produce proofs by deductive reasoning.

Introduction

For generations proofs were considered as tools for verifying mathematical statements and showing their universality. Hanna (1990) mentioned Leibniz who believed that "a mathematical proof is a universal symbolic script which allow one to distinguish clearly between fact and fiction, truth and falsity"(p.6). According to this approach the two classical roles of teaching proofs were to teach deductive reasoning as part of the human culture to be learned by human beings, and deductive reasoning as a vehicle for verifying and showing the universality of geometrical statements whereas experimenting, visualizing, measuring, inductive reasoning, and checking examples, were not counted for this purpose.

Recently there has been a change in this approach; it seems that there is a consensus that deductive reasoning -- proving -- still has a central role in geometry learning. However, the classical approach is now enriched by new facets and roles. The most important contribution is due to the development of dynamic geometry software such as the Geometry Inventor (Logal, 1994). A main pedagogical feature of many dynamic geometry based learning environments is that students are partners in the discovery of geometrical facts and in the reinvention of geometrical relationships, by means of exploration and inductive reasoning (Goldenberg & Cuoco, in press).

As a consequence of changes in mathematics and mathematics education, which where amplified by the existence of these tools, two roles of proof will be discussed here: **Convincing** - deductive reasoning as a vehicle for convincing oneself and others that a geometrical statement which was rediscovered or conjectured is true, and **explaining why** the rediscovered conjecture is true (Dreyfus & Hadas, 1996).

Recently, it has been argued that when Dynamic Geometry software is used in teaching geometry, conviction can be obtained quickly and relatively easily (by induction and experimentation), and therefore proof will become less important. For example, de Villiers (1997) argues that in learning with or without dynamic tools, convincing someone of the truth of a conjecture should take precedence over the process of proving, and the only important role of proof in a dynamic environment is as a tool for explaining.

We claim that, by careful design, based on experimentation and cognitive analysis of students' actions, situations can be constructed where students will had a need for proof, both for conviction and explanation: (i) **Conviction** by explanation in situations where the findings themselves, even while working in a dynamic geometry environment, are not convincing. (ii) **The need for explanation** in cases where the findings are convincing but surprising, and the surprise causes the need to understand why.

In this paper, we will exhibit an example of the development process of such an activity which has several cycles of design, experimentation and analysis. We will describe the work of two pairs of students in two different cycles of the development, and analyze the task as well as the work of the students. Finally, we discuss principles of activity design for leading students to feel the need for and to produce proofs by deductive reasoning.

The activity

About 6 months into a geometry course with the use of the Inventor, two pairs of 9th graders, worked on the following activity:

Task 1. *Which values can the base angle of an isosceles triangle have? explain!*

Task 2a. *Follow the median and the angle-bisector from the same vertex, in a "dynamic triangle". What can you say about the triangle when both segments coincide?*

Task 2b. Draw *the median and the angle-bisector from the other two vertices. Try to find a situation where two pairs coincide and the third pair does not. Explain!*

Task 3. *Divide the side AC of a dynamic triangle into 3 equal segments. Connect the division points to the vertex B* (Fig. 1).

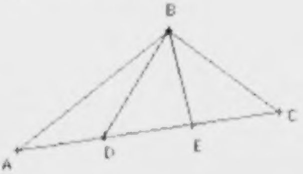

Figure 1

Investigate the relationships between the sizes of the 3 varying angles (<ABD, <DBE, and <EBC). Explain!

Interviews with two pairs of students working on this activity were videotaped and analyzed. Here we will discuss mostly the third task, for which the first two form part of the "pedagogical history". It is worth noting that in a previous lesson the students were engaged in another activity in which they concluded that if two triangles are equal in two sides and the angle opposite one of the two sides, then either the triangles are congruent, or the sum of the two angles opposite the second side equals 180^{o}.

The interviews

First pair of students

In task 1, Tammy and Shiri, two girls, concluded that the size of the basis angle in an isosceles triangle varies between 0^{o} and 90^{o}. In task 2a. they concluded that when the median and the angle bisector from a vertex coincide, then the triangle is isosceles. After drawing Figure 1, they soon guessed that the three angles have to be equal. Using the measuring tools on the 3 angles, they were surprised for the first time. The following are some excerpts from the interview (I= interviewer):

I: *Try, if you can, to find situations where the 3 angles are equal and characterize these situations.*

The girls changed the triangle by dragging and watched the change of the angles visually as well as numerically.

Shiri: *There is a situation where all of them are equal.*

Tammy: *When A and C* (the vertices) *are moving apart the middle angle takes all the angles* (she means degrees) *from others two.*

I: *What will happen if you will drag only A?*

Tammy: *Then the outside angle* (<ABD) *will become very small and will not be equal to the other two. May be when <B will be very small the angles will be equal* (she dragged B until the triangle shrunk to a segment). *Oh, but it is not a triangle anymore.*

Tammy who relates consistently to the process of the visual change she creates, accompanies all her claims by hand movements.

Shiri, who was observing Tammy's actions quietly, interferes: *The three triangles can't be congruent.*

I: *Why?*

Shiri: *Let's take an example where the two angles are equal* (points on <ABD and <DBE) *than AD is the angle bisector and median as well, and ΔABE is isosceles. If the other two angles are equal as well than ΔDBC is isosceles as well, and then the three triangles are congruent. In this situation we will have 6 equal angles* (the marked angles in Fig. 2)*.........but, we saw before* (Task 1) *that the base*

angles in isosceles triangle are always less than 90^o, *and here the two are adjacent angles and this is impossible situation.*

The two girls are discussing together, Shiri explains again her claims to Tammy and writes down her explanation.

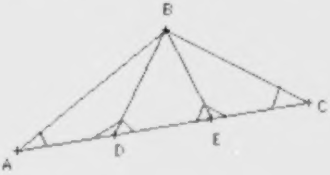

Figure 2

We point out two main issues here: The students' need to be convinced by an explanation and the resources they make use of in their explanation. The students' need to be convinced arose from their surprise when obtaining unequal angles and, even more so, by their failure to find even one situation where the angles are equal. The fact that it is impossible to check all cases made the students feel the need for general considerations whether there is a situation with three equal angles, and if not, why not. Tammy, who was more active at the beginning, tried use the dynamic tool to visualize extreme cases. When both girls saw that they cannot get to a final conclusion by working with the software only, Shiri, the silent but very involved partner, took over and began to propose deductive arguments. The resources for her deductive explanation were the history of what they did in the previous two tasks (and in earlier activities). Another very crucial point in this situation is that the students were fully convinced only by the deductive explanation--proof -- even if it is not written formally, and if not all details are specified.

Second pair of students.

In task 1, Tommer and Lior, two boys, saw immediately (without using the software) that the size of the base angle in isosceles triangle is changing between 0^o and 90^o. In task 2a, they used the software to explain that one gets an isosceles triangle when the median and the angle bisector from the same vertex coincide: They did it by reflecting the triangle.

The interviewer presented task 3 by drawing a sketch of figure 1. Lior thought that the three angles were equal, and Tommer responded: *"Why? It does not have to be like that because if triangle ABE is not isosceles, than BD the median does not have to be an angle bisector."*

The boys then used the software to find situations where the three angles are equal. They conjectured that this will happen when the triangle is equilateral and they checked their conjecture with the software. When they found that the angles are very close (19.1°, 21.8°, 19.1°), but not equal, they returned to the general triangle to look for cases in which the angles are equal. The interviewer suggested to use the graph option and asked: *Suppose we will draw three graphs, describing the change of each angle when AC/3 (the opposite segment) is changing , in the same coordinate system. What on the graph will show situations where the three angles are equal?*

Note: In the graph option, the graph of the changing variables is drawn in real time, while dragging the geometrical shape.

Tommer replied: *This will happen when the three graphs will have a common point.*

Guided by the interviewer the boys built a triangle were AB=4 and <A=30°. While drawing the graphs they guessed and discussed the characteristics of the graphs. After drawing the first two graphs, (See Fig. 3), Tommer claimed: *There is no need to draw the third graph ,because it will pass through the common point of the first two. So, we have the situation we look for.*

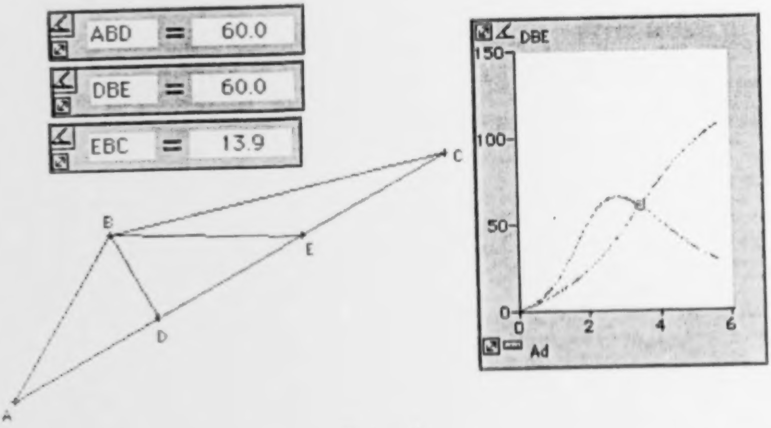

Figure 3

The boys did not looked at the figure itself, nor at the measurements they had, to confirm or refute the above. However, Lior continued with drawing the third graph (Fig. 4), which "surprisingly" did not passed through the common point of the other two.

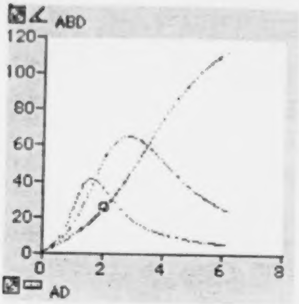

Figure 4

They concluded that they will never have three equal angles, and started to look for explanations, in spite of the fact that the Interviewer remarked that they had investigated a special case of triangles only. Lior dragged the triangle to match the intersection points of the graphs, and started to speak about congruence triangles, but was stopped by Tommer who exclaimed: *I know, I know!* and explained: "*If DB is both median and angle bisector in ΔABE, than the triangle is isosceles and BD is a height as well, and the same for BE in ΔDBC, and we cannot have two perpendiculars to AC from vertex B.*

It seems that Tommer realized that each intersection point expressed a situation of an isosceles triangle.

Lior, like the two girls above, was expecting the 3 angles to be always equal, while Tommer realized from the beginning this is not necessarily the case. But Tommer was also convinced that there are situations where they are equal. The fact that the three angles in the equilateral triangle are very closed (19.1°, 21.8°, 19.1°), supported their intuition that they can find a case of equality, and therefore Tommer even suggested not to draw the third graph.

The real surprise, in this interview, arose only when the boys realized that the three graphs do not have a common point. In spite of the fact that they were already convinced that the three angles will never be equal, they felt the need to understand why and to explain it deductively. Tommer's resources of explanations resulted from the action of matching between the different representations of the intersection points, and like Shiri's, from previous tasks, in particular task 2.

Conclusions

The goal of this paper is to show some relationships between the design of an activity with certain pedagogical purposes, and cognitive research on students' actions in this activity.

We started from a pilot version of the activity designed, according to our beliefs and experience, to fit the learning of proofs as a tool for explanation and conviction. This pilot version formed the basis for the first interview. The analysis of the interview lead us to a second version of the activity, according to which the second interview was planned and carried out, (as well as analyzed in view of a third version of the activity). We will now summarize the first two cycles, the final version (at least for now), and some global principles for the design of activities for meaningful learning of proofs in geometry with a dynamic tool.

As expected, the activity in its first design resulted in a surprise for the students. This surprise caused the need for an explanation in order to understand the surprising findings, and to be convinced of their truth. Clearly, the two girls in the first interview were not satisfied by visual considerations, while dragging the vertices and changing the triangle, and therefore moved quite smoothly to deductive explanations.

We decided to enlarge the investigated problem situation by inserting an additional tool -- the graphical representation of the varying angles. In this we had a twofold goal: (i) To create an additional potential source for explaining and/or convincing.

(ii) To expand the students' conceptions of the variables, the way they are varying, and the relationships between the geometrical, the numerical, and the graphical representations of this variation.

The graph option in the second version, was tried in the second interview. Although it had a great power of conviction for these students, the need for explanation arose even more explicitly. It is worth to note that the matching between the intersection points and the corresponding geometrical situations was an additional source for constructing students' explanation, which in the end was built deductively.

We decided to add another item in which we will try to persuade students to check the graphs for additional cases of the givens, for example, to attempt to make the three intersection points close by choosing a larger value for <A. After checking the 3 graphs for few similar situations, we expect students to realize that such checking can not solve their uncertainty, and that only the understanding why the three angles will never be equal has the power of convincing. Goldenberg, Cuoco, and Mark (in press) said that "a proof, especially for beginners, might need to be motivated by the uncertainties that remain without the proof, or by a need for an explanation of why a phenomenon occurs. Proof of the too obvious would likely feel ritualistic and empty". There is no doubts that a situation like in the above activity, where one can not find any example for her/his conjecture, has the potential to create uncertainty of this kind.

In conclusion we would like to sum up global features of "activities that induce the need for explaining and convincing" which were demonstrated by the development of the above activity.

The need for explanation was raised by:

- a surprise caused by the contradiction between the conjectures and what students got (or could not get) while working with the dynamic tool.
- a situation where one can not find any example for a conjecture he or she made.
- the multiple representations of the situation (geometrical, numerical and graphical).

The resources for explanation and conviction. were constituted by:

- Conclusions based on previous tasks. The activity is planned as a sequence of interrelated from a contextual point of view, as well as by their potential to serve as resources for explanations .
- The possibility to have analogies in the three representations: in our activity the numerical and geometrical meanings of the intersection points of the graphs.

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The case of Rita: "Maybe I started to like math more"

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Abstract

This article is a case study of the dynamics of attitudes of one seventh-grader. First I present her negative attitude towards mathematics ("stupid", "You don't need math in life"), then how a negative emotion develops in a problem solving situation. From insecure beginning it changes through frustration to rejection. She works in a group and this emotional process is connected with a social process. I suggest that "I don't need this" is her defence strategy, and that similar emotional experiences are a reason for her negative attitude. Within half a year her attitude towards mathematics changed drastically ("mathematics is quite nice"). The reason she gives for this change, is that she has "been understanding it a bit more".

Previous research and theoretical background

Mathematics is a school subject that many pupils have quite emotional relation with. Satisfaction and joy often accompany a successful solution of a problem (McLeod, 1988). On the other hand many hate mathematics, even panic about it (Buxton 1981).

On classroom level the general attitude of the class towards mathematics is clearly related to teacher quality (as perceived by the pupils). During the junior high school the social-psychological climate of the class is also becoming an important determinant of class attitude. (Haladyna et al., 1983)

The dynamics of attitudes are not well understood. Attitudes tend to become more negative as pupils move from elementary school to secondary school. Efforts to promote desired affects of students have usually induced only slight changes and sometimes even contrary to expectations. (McLeod, 1994)

Some success has been achieved. Mathematics anxiety has been reduced through a psychological treatment called systematic desensitisation (Hembree, 1990). Nation-wide policy in Australia to promote girls' participation in mathematics has been successful (Blackmore et al. 1996).

McLeod (1988) sketched a theory of affective issues in a problem solving situation. He suggested the following aspects to be studied: the magnitude, direction, duration, level of awareness and level of control of the emotion. Qualitative approach has given us new insight in this field. Hart and Allexsaht-Snider (1996; in McLeod, 1994) brought up the notions of "belongingness" and "resistance" as central aspects to study of motivation. Goldin (1988, in DeBellis and Goldin, 1997) presented "affective pathways" as a structure for the dynamics of affective domain in mathematics. These pathways are established sequences of states of feelings that interact with cognition and suggest strategies during a problem solving process.

DeBellis and Goldin (1997) presented a model, where four components interact on individual level: emotional states, attitudes, beliefs, and values/morals/ethics. Interaction with environment is also included in the model.

Research project and the focus of this report

This research explores how pupils' beliefs about and attitudes towards mathematics are changing through their lower secondary years (grades 7 to 9). A type of action research is fitted to the project, in which the author acts as a teacher researcher: one class will be taught mathematics for these three years (August 1996–May 1999) with an attempt to implement girl-friendly pedagogy. Pupils will be observed in school, and pupils, their parents, and their other teachers will be interviewed.

This report focuses on the dynamics of attitudes of one particular pupil. This case is especially interesting, because Rita's attitude has changed to desired direction.

Methodology and data

I work with enactivist methodology, where the two key features are "the importance of working from and with multiple perspectives, and the creation of models and theories which are good enough *for*, not definitively *of* " (Reid, 1996, p. 207). Qualitative approach was chosen to understand the dynamics on individual level.

Large variety of data on Rita was available for me as her teacher and form master. I have tried to reach the multiplicity of perspectives through discussions with Rita's other teachers and fellow researchers. I have reviewed the material several times and selected some parts that I felt to be relevant. Theories from women's studies and beliefs research have helped to understand the case.

This paper relies mainly on two interviews (December 1996, May 1997). Some episodes that were recorded in my diary and a few lines from a third interview (December 1997) will be used too.

The code-key for the transcription:

(x.y); (.)	pauses: x.y seconds; less than 0.5 seconds
(-); (--); (*text*)	unclear speech: one word; several words; possible words
wo(h)rd	word has been spoken laughing
[*text1*]; [*text2*]	texts 1 and 2 spoken simultaneously
=	talking continues immediately after the other speaker
{text}; {...}	editorial comments: about context or tone of voice; text omitted

"You don't need math there in life"

At the time of the first interview I had been teaching the class for four months. Rita didn't do homework regularly and she was occasionally disturbing the class. I evaluated her success satisfactory. In this interview there were also two other girls from the same class. Rita's comments were not too flattering for me as her teacher.

[13] Rita: *Mmmm. It* {mathematics} *was nicer in elementary school than in secondary school.*

[25] I: *Describe me a lesson that you remember from elementary school.*
[31] Rita: *Help.* (.) *errr* (.) *I don't remember anything but at least we had so, that when we had then the multiplication table* {...} *And alike. I don't remember anything, it was so stupid.*
[38] I: *What has been most boring?*
[39] Rita: *The, thethethethe story problems.*
[40] Maria: *I think that story problems are* [*sort of...*]
[41] Rita: [*I don't understand*] *them* [*ever*].

Maria continues telling how she prefers story problems over routine tasks. Rita changes the subject.

[49] Rita: *You don't need math in life. I think. Because I do know enough math to manage when I go to buy a shirt or need to know the time or such.*

The other two girls insist on mathematics being useful in different professions and in shops. Rita assures that she can do all that. The discussion goes on.

[89] Rita: *I can't explain, but in a way like* (1.0) *now when we have really strange things in math. All that we have had at elementary school, all fractions and such, and these you do need, but not these* (1.1) *these other things.* (1.6) *These, I can't explain, the things that come for example on ninth and at high school. You don't much need those there in life.*

In Rita's last sentence the word *'there'* reveals how she sees school life alienated from real life. In the real life out there, she doesn't need the mathematics that is taught in school. She already masters what she needs.

***"You don't need this"* is a defence strategy**

In the first interview I gave the group of girls three tasks to solve together. I gave the written tasks one at a time and recorded their solving process. Here I shall summarise the process and concentrate on Rita's contribution.

Task 1. Salla is working on an abstract painting. She has divided an area with straight lines into parts. She wants to paint the picture with as few colours as possible. Parts that are side by side, may not be of same colour, but those touching only in corners may. How many colours will Salla need. (Below the text was a picture that could be coloured with three colours.)

Maria and Lisa start solving the task together, trying to find out one possible colouring. Rita's comments are few, and she gets no response. From the discussion I extracted here all Rita's statements and some discussion where Rita is interacting. The running time is shown on the right side.

[277]{Beginning the task} {0.00}
[293] Rita?: [(-*yellow*)] {0.43}
[308] Rita: *I don't like this t(h)ask at all.* {1.15}
[325] Rita: *Yhm. Yes* {2.00}
[327] Rita: *Is this then yellow, 'cos that (-).* {2.03}

[332] Rita: *How come it's blue then?* {2.18}
[333] Lisa {to Maria}: *Yes, probably it would go with three colours.*
[334] Maria {to Lisa}: *Three colours.*
[335] Rita: *Hey! Because that one is yellow.* (3.0) *Mm?* {2.20}
[336] Maria {puzzled}: *What did you say?*
[337] Rita: *Why you put that blue?* {2.30}

Rita gets an explanation and thereafter continues together with the other two.

Rita seemed to have difficulties in the beginning. I had to tell the girls to move closer to another so that Rita could read the task. At the beginning of the solving process she got very close to frustration. After two minutes there was the first sign of understanding. When she tried to break in, the other two seemed to ignore her first, but she was persistent and was taken in.

The second task was an estimation of letters in a given book. Rita understood the task at once and was an active contributor in the solving process.

Task 3. Addition, subtraction, multiplication and division are operations. Let's define a new operation # in a following way:

When a and b are numbers, then $a\#b = (a+b)*(a-b)$.
An example: $2\#3 = (2+3)*(2-3)= 5*(-1)=-5$

a) Do the following calculations:

$2\# (-3) =$
$(-2)\#3 =$
$(-2)\#(-3) =$

b) In addition you may change the order. For example $2+3=3+2$.
May you change the order in the defined operation #?

This was a difficult task and there was 16 seconds of silence after they had read the task. The beginning was similar to the task 1, where Rita was left as an outsider as the other two did the solution. In this case however her persistence didn't last long enough.

[486] {Rita's last effort to contribute} {0.00}
[500] Rita {tired voice}: (*That is*) *a nice* (*task*). {0.50}
[508] Rita {half yawning}: *What would have been the right answer?* {1.15}
[513] Rita {offers chewing gum}: *You want some?* {1.24}
[515] Rita {checks if she has more}: *Let me see.*
[518] Rita: *I don't understand a piffle of what they are even trying to do the(h)re.* {1.42}
[528] Rita {parodising Lisa and Maria}: *Minusminus five minus minus six hundred.* (1.7) *Look, you don't need this for example in your life.* {2.40}
[530] I: *Yhm.*
[531] Rita: *These are exactly the kind* (*I mean*).

Here Rita claims that this kind of mathematics is not needed in life. However, that does not seem to be the reason for her giving up work. **First** she tells, that she doesn't understand. **Next** she taunts the task. **Finally** she tells, that this kind of mathematics is not needed.

Rita's process fits with what Goldin (1988, in DeBellis and Goldin, 1997) calls 'a negative pathway'. There frustration turns into anxiety, and despair and these emotions evoke defence mechanisms, avoidance, and denial. You can also see how emotions (frustrated) awaken a belief in mathematics (*"you don't need this for example in your life"*). There is a link to values: this mathematics has no value.

We may also understand Rita's actions using the terms "belongingness" and "resistance" (Hart and Allexsaht-Snider, 1996; in McLeod, 1994). Rita is willing to work with her peers. The other two don't seem to notice that Rita can't always follow. With the first task she was persistent and was able to break into the discussion. In the third task she remained outside and finally took a resistance-position.

Could it be, that telling that some kind of mathematics is not needed actually is a defence strategy of the self? Being left outside is not easy for Rita. As she found out that she couldn't follow, she tried other approaches. She started to make ironical comments on the task. As the others ignored that too and even seemed to enjoy the task, she probably felt more and more rejected. So she made a counter-attack to reject the task.

Rita's position of resistance can be seen also at her comments when I asked the girls how they liked the tasks. Rita thought the two first ones were OK, but the third task...

[558] Rita: *That was really stupid. You don't need such in life.*
[559] Maria: *I liked to do that one especially.*
[560] Rita: *You will certainly become some philosopher (-) when you grow up.*

Another episode from the next fall (Diary, 24.9.1997) reveals how this defence strategy is linked with understanding. A friend argues that she doesn't need powers and Rita replies - not claiming the need - but the easiness.

> Pia: *What we need these powers for? {...} I don't need these.*
> Rita: *These powers are really easy.*

"I think that now mathematics is quite nice sort of"

Rita did well in the next mathematics test and was very happy about it. She commented it several times, telling that this number was the best she had ever had in a mathematics test. In April she asked "What we need this geometry for?". Altogether she was more active during the spring than she had been during the fall. Her success in geometry test was poor, however.

In May, at the end of school year my class filled in a questionnaire. One of the items was about the way they think about mathematics now compared to last year (Table 1). Most pupils' views of mathematics had changed during their first year at secondary school. Rita was one of the two who stated the most drastic change.

Response	-3 (totally disagree)	-2	-1	0	1	2	3 (totally agree)
N	2	4	4	3	1	0	0

Table 1. Pupils' responses to the statement: "I think of mathematics the same way I did last year"

I had individual interviews with pupils after the questionnaire. I asked Rita to explain her answer to this question.

[153] I: *Well here you had a strong opinion that you think differently about mathematics compared to last year.*

[155] Rita: *Yes. Sort of now sometimes mathematics has been a bit more fun, because been understanding it a bit more. I have always had a six* [the second lowest to pass] *or something in math, so I've been a bit more along now.*

[158] I: *Why so?*

[159] Rita: *I don't know, maybe I have started to like it more. I think that now mathematics is quite nice sort of. In elementary school I didn't like it at all.*

[161] I: [...] *why math is more fun on secondary than on elementary level?*

[163] Rita: *I don't really know. Maybe if I have learned more sort of tho... things sort of, so it is easier to do it. I don't know.* (3.0) *Hm.*

Rita is clearly aware that she thinks differently about mathematics. Yet she doesn't know why this change had happened. The reasons she gives seem to be in circles: It's *more fun* because she has *been understanding more*, because *mathematics is quite nice* because she *has learned more*.

What she said in the second interview about elementary school seems to contradict her statements in the first interview (*"It was nicer in elementary school"*). At that time neither of us remembered that, but I pointed these contradictory statements to her later (23.9.1997). Rita was astonished: *"Oh, really? It must be because now I have begun to understand better"*. So she does not only understand now better than in elementary school, but also better than at the time of the first interview.

Earlier in this second interview she had told about growing as a person.

[27] Rita: *I think a bit more what I do. And why I do.*

[28] I: *Could one say that you have become more (.) considering?*

[29] Rita: *Mmm. And I have sort of taken more responsibility on my own actions.*

She also had told that she wants to have a good profession and that next year she needs to spend less time with friends and study instead. Her career aspirations, however, are the same (not at all mathematical) as in December. I tried to connect her grown responsibility to mathematics learning. She doesn't accept this interpretation, but holds on to her own.

[168] I: *So has it come just because you have done a bit more work. I mean worked more conscientiously=*

[170] Rita: =[*Yes, or*]

[171] I: =[*And so*] *you have learned more, or is there something else involved?*

[173] Rita: *Yes, or maybe I have been in a way more been a bit sort of interested. Or have been more in a way I don't really know (.) In a way.*

I take her by the word and ask her what she has liked in mathematics. This seems to be a difficult question for her to answer. She had to think for some time before she was able to squeeze out an answer.

[181] Rita: (3.0) *Hm* (1.5) *I don't know, sort of in a way* (1.2) *sort of* (2.0) *that you need to sort of* (.) *sortofmmh* (1.9) *I don't know, mh, oh no, so that one needs to little* (.) *think over sort of* (.) *such problemstasks.* {...} *Although it doesn't go right, it is still sort of* (.) *nice to think and even if it goes wrong, so it is nice to know how was it right so that next time could do it right.*

She told to like problems that require thinking. This statement was, however, far from spontaneous. Maybe she felt that she had to give some explanation. Because she couldn't specify any task she had liked, I didn't regard this statement reliable. Later in the interview, however, she suggested group projects for next year, where *"drawing and considering, drawing and considering"* would be needed.

The next fall, after summer holidays Rita has been active in mathematics class and eager to learn.

Diary (2.9.1997)
{Rita was very active today.}
Tina: *Rita, you haven't been active in the class before.*
Rita: *Yes, I have, haven't I?*
I: *You have been participating, but I don't recall you being this active last year.*
Rita: *Good. Then I will get a nine* {the second best grade}.

In Rita's mind, liking seems to almost equal understanding. This can be seen in her comments from fall 1997.

Rita: *That* {powers} *is easy! That mu(h)st be the nicest thing exactly that one understands the topic.* {third interview, December 1997}

Diary (23.10.1997)
{Introducing irrational numbers. Rita is having difficulties.}
Rita: *Now math is becoming stupid again.*

Diary (27.10.1997)
Rita: *We don't need these* {*irrational numbers*} *anywhere, do we?*

Old habits die hard. As Rita is having difficulties with this topic she seems willing to fall back to her old defence strategy of rejection. But she does not. On the contrary. Although several weeks had passed since we needed irrational numbers, she told in third interview that she wants to understand this topic, too.

Discussion

This report illustrates how emotions in a problem solving situation shape beliefs and values of Rita. Frustration and despair led her to label a task useless. How far can we generalise this? On more general level she disvalued mathematics. I am tempted to believe the reason is repeated lack of understanding? If we accept this one case, why can't this happen with some other pupils too, at least sometimes. Often pupils in the class ask "what we need this for?" The teacher should take it as a warning sign. Maybe they do not understand what has been taught?

The second half of this story gives us hope. A drastic change, almost falling in love with mathematics was a simple process. It took only half a year and no special treatment was needed. But it is not simple. She has classmates whose attitudes have followed the more common line towards disliking mathematics. I haven't been able to distinguish any critical factor. Understanding was the key concept for Rita. Why did she understand, while some others didn't? Was it the test she did well or her more serious attitude towards school? What was the role of pondering problems? I believe the right cocktail of this all was necessary.

This story was a pupil's story for her teacher. I tried to present it as authentic as possible, but the question of bias can not be ignored. Your task is to judge the relevance of this story. In Masons (1994, p. 184) words: "The test of validity is whether it generates convincing stories about the past, and whether it informs actions in the future".

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ON TEACHING EARLY NUMBER THROUGH LANGUAGE

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In an inner-city secondary school in Birmingham, 14 students aged between 11 and 14 years of age were identified as having considerable problems with mathematics, partly due to a lack of confidence with number. These students were exhibiting many errors in their writing and reading of numbers which are common amongst children half their age. This paper looks at some developments in our own understanding of the issues involved as we devised and implemented a number recovery programme using a language based approach to early number work.

Gray and Tall (1994) have talked of a proceptual divide between those children who rely upon procedures, such as count-on, and others who use meaningful number facts, such as $7+2=9$ so $70+20=90$, as well as procedures (proceptual thinkers). Two of our students, Tim and Debbie, illustrate this difference. Tim would consistently use a counting-on and counting-back procedure no matter which numbers were involved in a problem. For example, he took 2 minutes 52 seconds to answer $67 - 41$ incorrectly by mentally counting-back on his fingers. Debbie, on the other hand, could use flexible methods involving complements to 5 and 10. For example, when correctly solving $6+7$ she said: *add the two fives and then add the two and the one and then added them.* Such differences in approaches are classified within various models used by Cooper, Heirdsfield and Irons (1996); Foster (1994); Wright (1994).

Despite Gray et al's (1997) warning: *we conjecture that positive efforts to make the relationships implicit in proceptual thinking explicit to those that do not have the associated flexibility run the danger of being seen by some as a new set of procedural rules* (p 121), we wanted to consider ways in which we could help students develop more flexible approaches. Gray et al (ibid) also conjectured that *one cause of the proceptual divide is the qualitatively different focus of attention which, on the one hand places the emphasis upon concrete objects and actions upon these objects, and on the other on abstraction and the flexibility intrinsic within the encapsulated object* (p121). We decided not to work with concrete objects but to stay with the language of number words and thus work metonymically, with the symbols (both written and spoken) rather than any meaning in terms of cardinality. Indeed we decided to reject such metaphorical tools as Dienes' blocks for place value, instead staying with the metonymic structure inherent within the numerical symbols and their relationship with each other. Cobb and Yang (1995) observed that 'place value' was not taught as a separate topic in Taiwan, and the metonymic approach we are taking can also make this an unnecessary topic (Tahta, 1991).

In this paper, we describe some of the teaching strategies used, and the reasons behind the decisions taken, within a short four week recovery programme with one 50-minute and three 10-minute sessions per week. The focus for the programme was to help students build confidence in being able to say and write whole numbers, and develop mental strategies for adding and subtracting whole numbers which encourage alternative approaches to those of count-on and count-back. We identify some of the factors and incidents which have led us to develop our own awareness of issues involved in helping students who, despite 6-8 years of schooling, have failed to gain a confidence with number. Thus, this paper is concerned with tracking our own awareness rather than reporting on the progress of the students involved, and as such we are working under the Discipline of Noticing outlined by Mason (1994).

WORKING WITH WHAT IS AVAILABLE WITHIN THE LANGUAGE

Regularising the language

The difficulties inherent within the irregularities of English and European number names as opposed to the regularity of Chinese and Japanese are well documented (see, for example, Miura & Okamoto, 1989; Fuson & Kwon, 1991). An issue for us was how we would deal with such irregularities when we were going to work so closely with the spoken language of number names along with the written symbolism of number. Wigley (1997) points out that *significantly, the greatest irregularity* [in English] *is in the second decade, so that learning numbers in the natural counting sequence does not help understanding of place value at that crucial point where it is first used!* (p114). The multiples of ten are also irregular and Fuson and Kwon (1991) discuss the possibility of creating a 'Chinese' version of English where the word 'ten' is used (12 would be read one ten two). We decided to work with an alternative, regular, spoken language using one-ty (10), one-ty one (11), one-ty two (12), one-ty three (13), etc., and two-ty (20), three-ty (30), four-ty (40), etc. for the decades. This is suggested by Gattegno (1974) and has the benefit of fitting in with the regularity of the higher decade words: six-ty, seven-ty, etc.

Students were introduced to the regularised language through whole-class counting tasks, such as counting up and down in ones, fives and tens, now referred to as "tys". We also used this language in conjunction with the tens chart (Gattegno, 1974, 1988) (see Figure 1), pointing at individual numbers on the chart and asking students to say them. The changes made in our spoken language of number names became increasingly apparent in the language spontaneously used by students and they became comfortable with the idea that one number can be said in two ways:

EB: What number is that ? (points to 19 written on board)
Jo: *Nineteen*

EB: How else can we say it?
Students: *One-ty nine*

1000	2000	$\boxed{3000}$	4000	5000	6000	7000	8000	9000
100	200	300	400	500	600	700	800	900
10	20	30	$\boxed{40}$	50	60	70	80	90
1	2	3	4	5	6	7	$\boxed{8}$	9

Figure 1: Part of the tens chart, with the component parts of 3048 highlighted

Reflection on 'Regularising the language'

In commonly used spoken language, the awareness of twelve as one ten and two is not available in the language, whereas this awareness is available if 12 is spoken as one-ty two (once the learner is familiar with -ty as ten). Thus, what is available to the learner solely from within the language changes as a consequence of the regularisation of the language. The structure of the tens chart reflects what is available in the regularised language. The digit words are represented horizontally (*one, two,...*), with the value words represented vertically (*ty, hundred, thousand*). Thus each component of a number can be thought of as a digit and a value.

In the question, "*how many hundreds are there in five hundred?*", the answer is clear from within the language. This is also the case with "*how many tys are in five-ty?*" as opposed to "*how many tens are there in fifty?*". The regularised language helped us phrase questions to students in a similar way whether we were discussing hundreds, thousands or 'tys'. It was also helpful for the students to hear the value of the tens digit within what they and their teacher was saying. However, there is a potential cost for the student in learning and using a new collection of number names which are specific to the social discourse of the classroom. Our experience so far is that students have adopted the new names quite readily but we are unsure at present whether any awareness gained from using the regular names will also transfer to the use of the normal irregular names.

A lot from a little

A guiding principle for us was to help students maximise what they could do from the skills they already possessed. We used stressing and ignoring as a teaching strategy to draw students' attention to what is available within the language. For example, having established with the group orally that four plus three is seven, we would ask what **four**-ty plus **three**-ty, **four** hundred plus **three** hundred, **four**-ty thousand plus **three**-ty thousand would be. In a similar way we stressed parts of the spoken language to solve sums involving, for example, 3 + 4 = 7, such as 15**3** + **4**, 46**3** + **4**, 786**3** + **4**. We extended these methods using language and the tens chart to work on a wider range of sums. The students were generally able to add a single digit to another single digit (whether by counting on or through more flexible approaches). Thus, sums such as 425 + 300 could be done by doing single digit additions within the hundreds row of the tens chart. Likewise, 425 + 50 and 425 + 1 each involves addition in only a single row of the chart. Then these were combined to do all three together, 425 + 300 + 50 + 1, and written as 425 + 351. Despite the fact that most students were still at the stage where 4 + 3 = 7 was not available to them as a known fact, all bar one of seven students attending the final fifty minute number recovery lesson (the exception being someone who had attended less than half the sessions) attained an average success rate was 84% doing such calculations as 804 + 4124 mentally. Many of these students had previously been put off from even attempting such calculations because the numbers appeared too big to them.

Reflection on 'A lot from a little'

A key issue for us within our own development is to identify which skills can, and which cannot, be developed from existing skills. For example, the above additions were carefully chosen to build on existing skills so did not to involve the new skill of 'going over the top' where a 'carry' is involved. At present we are considering ways in which we would involve the tens chart to help work on what happens when the addition involving digits of a certain value affect the digits of another value (e.g. 3127 + 192: the addition of the 'tys' will affect the addition of the hundreds). While the students still needed to develop their knowledge in areas that cannot be worked on through the language, such as number bonds and partitioning, our work suggests that a lack of such number skills does not have to be a barrier to developing flexible skills for mental addition of larger numbers.

MAKING EXPLICIT WHAT IS HIDDEN

All of the students, at various times, demonstrated errors when reading and writing numbers with zeros in. Responses to writing down the number *six thousand two hundred and fifty-one* included 600251; 620051; 60020051; and 60251. There are clearly issues here for students about when and where to write zeros. The zeros are hidden both within the symbolic number notation and within the language. When writing *four thousand three hundred and seventy six*, the three zeros in four thousand (4000) are hidden in the final number 4376. A zero appears if the number changes to four thousand and seventy six, 4076, yet the zero itself is not spoken.

From static to dynamic images of numbers

The tens chart represents numbers in terms of their components, e.g. 4376 will be represented by highlighting 4000; 300; 70 and 6. The computer programme *Numbers*[1] allows students to see written numbers as the result of a dynamic process: the components of the number highlighted on the chart drop down the screen to become written as 4376. 4000 drops first, then the 300 fits over the top of the 4000, giving 4300, the seventy then drops down to create 4370 and finally the 6 falls into the units column leaving 4376. Our students practised writing numbers by asking the computer to highlight the components of a random number on the tens chart, writing down what they thought the number would look like written down and then checking by ‘sending down’ the components on the computer screen.

Within *Numbers*, once a number is written as a single, static representation, any digit in the number can be highlighted and it will rise up above the number showing its component value. For example, if 3 is highlighted within 4376, it will rise up as ‘300’, showing the zeros that are hidden in the static representation:

4376 Mouse clicks on the digit ‘3’	300 ↑ 4076 Digit ‘3’ rises up revealing hidden zeros and computer voice says *three hundred*	↓ 4376 Digit ‘3’ returns to original number

Reflection on ‘From static to dynamic images of numbers’

Teachers have often used cards which have component parts of a number written on them (see Figure 2). These can be put together to form a multi-digit number.

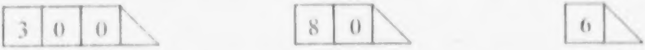

Figure 2: Cards of the component parts of the number 386

The computer program has an added element to such cards, as the component parts of a number can be seen as part of the tens chart, which represents the structure of the whole written number notation. The ‘risers’ mean that any digit within a number can be examined and the component part which it represents can be viewed. We differentiate here between a digit of a number, such as 8 in 386, and the component part which that digit represents, 80 in this case. The program also says the verbal number name for this component part at the same time as it ‘rises’: *eighty* or *eight-ty* - there are options as to whether the conventional names are said, or the regularised ‘-ty’ names are given. The movement in the computer program helps students to see the components within the whole number and how they relate. Several students showed evidence of their own internal dynamic processes. For example, when given 6441, shown in its components on the tens chart, Kay wrote 6440, then wrote a 1 on top of the 0. Andrew, when given the question 50042 + 3024 on his worksheet, wrote in a 3 on top of the first zero in 50042. The dynamic relationship between parts and wholes is one which we find very attractive as a pedagogic tool and helped students to see the value of a digit within the whole number.

Stressing that which is usually ignored

In Catalan Sign Language there is a specific sign or movement for zero in the middle of a number. This creates greater transparency between the written and spoken languages (Fuentes, M. and Tolchinsky, L., 1996). With the tens chart, the absence of a digit in a particular row can also be seen as the presence of an empty row in the chart. We decided to use a teaching strategy which drew students' attention to the empty rows on the tens chart as a way to helping them develop greater expertise in reading and writing numbers:

(The components of 4083 appeared on the computer tens chart)
Brian: *Oh God, I forgot to put...* (Brian had written 483 but now put in a nought to make 4083).
DH: (About half a minute later) You know when you forgot to put the nought? OK. It's no what? What have you got none of?
Brian: *Hundreds.*
DH: And have you shown that, where you put the nought? So what is it? It's no... We haven't got any what? Which row have we got nothing of?
Several: *Hundreds.*

It is possible to stress the presence of a zero in a number by saying it: 1076 can be said *one thousand no hundreds and seventy six*. Students showed evidence that they were making explicit the presence of zeros in a variety of situations, sometimes as suggested by the teacher and sometimes spontaneously:

- On board: 2052 + 500 + 20 = 2572
  DH: (Indicating the first number in the addition) I've got two thousand...
  Brian: *Two thousand no hundreds and five-ty two.*

- Andrew wrote 526 when the component parts of 5026 were highlighted on the tens chart. When the computer sent down the components to make 5026, Andrew saw his mistake he said *I forgot the hundreds*. Here Andrew shows an understanding that the zero represents something, i.e. no hundreds.

- When Kay was working on the task of writing numbers as the sum of their components, she seemed to represent the presence of the zeros as well as other digits:
  2003 = 2000 + 0 + 0 + 3
  2104 = 2000 + 100 + 0 + 4

Reflection on 'Stressing that which is usually ignored'

We feel that stressing an empty row in the tens chart has been useful in helping students become aware of the role the zero plays within written number notation.

Kay's use of zero, to explicitly represent *no hundreds* and *no tys*, interested us and made us consider how we would write the numerical equivalent of *no hundreds*: 000? And *no tys*: 00? Yet these both have a numerical value of 0. This raised for us an issue about the relationship between the verbal and written language.

Not only is it possible to stress what is available in the language to assist the learner, it is also possible for both teacher and learner to stress aspects which are usually ignored within the language in order to create greater transparency and overcome some of the difficulties with zeros. Hughes (1990) found that *young children are as adept at representing 'nothing' as they are at representing small quantities such as one, two and three ... It seems that this aspect of zero - representing the absence of a quantity - is not particularly difficult to grasp. Much more difficult is the use of zero in a place-value system* (p89). One reason we suggest this is problematic is because the zero is not spoken, and so is usually hidden, within the number name. Yet at the same time, the presence of a zero in the written notation is crucial. So, although we have attempted to regularise the number names, there is still not a direct correlation between the written and spoken language.

SUMMARY

We consider that language can play of fundamen al role in students learning about, and working with, number. Not only can the composition of numbers and the value of digits within numbers be worked on, but also addition and subtraction strategies can developed to offer greater flexibility for students. We have found that students can be encouraged to use larger units than ones (Steffe et al, 1983) in their counting strategies and so develop proceptual methods which are not just based on new rote learnt procedures, but are known through stressing the language inherent within number names. Through examining our own teaching strategies we are finding ways to help students exploit the language to help build their confidence and skills with number. In particular, we are developing our use of stressing and ignoring in different guises and in different contextual situations. We consider this to be an on-going process as we plan an extension to the number recovery programme.

The computer program *Numbers* is being developed by the Small Software Working Group of the Association of Teachers of Mathematics. Details can be obtained from Dave Hewitt who is convenor of the group.

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RELATING CULTURE AND MATHEMATICAL ACTIVITY: AN ANALYSIS OF SOCIOMATHEMATICAL NORMS

Lynn Liao Hodge and Michelle Stephan

This paper presents an analysis which relates students' and teachers' cultural backgrounds to their mathematical activity. We have used the elaborated interpretive framework of Cobb and Yackel (1996) to examine the negotiation of sociomathematical norms in two classrooms. We view individuals' participation in this negotiation as being reflexively related to how they participate in the practices of other communities, such as the school and community cultures. This analysis indicates a direction for future studies which attempt to examine the relationship between individuals' cultural backgrounds and mathematical learning.

An extensive body of literature exists describing how students' and teachers' cultural backgrounds relate to participation in the classroom (Au, 1982; Delpit, 1988; Phillips, 1972). Though these studies are helpful in emphasizing the importance of understanding the cultural backgrounds of students and teachers when examining events in the classroom, they are limiting in that they are content-free. They explore norms of participation which exist in all classrooms, but they do not describe how the cultural backgrounds of individuals relate to learning in particular content areas. The intent of this paper is to build on the work of these authors by relating students' and teachers' cultural backgrounds to mathematics learning. Specifically, the purpose of this paper is to show how the analysis of the negotiation of sociomathematical norms may be used to relate the cultural backgrounds of individuals to their mathematical activity. To this end, we will clarify what we mean by sociomathematical norms and how they relate to social norms. Then, we will discuss two classrooms which we would label "math-inquiry" in that the two teachers made it a point to negotiate social and sociomathematical norms with their classes. The analysis of the two classrooms offers an example of an analysis which relates the cultural backgrounds of students and teachers to mathematics learning.

Theoretical Framework

We use the interpretive framework of Cobb and Yackel (1996) as a means of conceptualizing the negotiation of social norms and sociomathematical norms within the classroom. The first level of their framework features classroom social norms. These norms include explaining and justifying solutions, making sense of the contributions of others, and indicating agreement as well as disagreement. The second level of Cobb and Yackel's framework deals with sociomathematical norms, which relate specifically with mathematical problem solutions. These norms include understanding of what counts as a different solution, a sophisticated solution, and an efficient solution. Also involved in the analysis of sociomathematical norms is how the teacher and students constitute what is considered an acceptable solution. The negotiation of social norms is important in that it contributes to the development of sociomathematical norms and the classroom microculture. If the social norm of explaining one's answers is not negotiated in the classroom, the sociomathematical norms of what constitutes an acceptable explanation or what counts as a different solution would clearly be difficult to negotiate. In addition, sociomathematical norms are different from social norms in that social norms can be found in any subject area. For instance, we would want students to explain their answers to questions in a history or English class. For this reason, we confine the analysis to the negotiation of sociomathematical norms since they are specific to mathematics. However, we do recognize that the negotiation of social norms is a critical aspect of the classroom microculture

Clearly, the events of the classroom are not isolated, but occur within a broader context. As Apple (1992) emphasizes, it is important to consider the classroom as situated in a larger context in light of issues regarding diversity. He warns that we must carefully examine the policies which structure inequity in the school context as well as the wider society. For these reasons, it is important to conceptualize the classroom within the larger context of school and society. We consider Cobb and Yackel's (1996) elaborated framework shown in figure 1.

[Insert figure 1 about here]

In this figure, the outer two boxes describe the norms and practices at the school and societal levels. In addition, the figure includes the practices of institutions and places individuals' activity within a context. In this way, the events of the classroom are affected by and in turn affect the practices of the school and society. In other words, the relationships among the classroom, school, and society may be viewed as reflexively related.

Figure 1 is also helpful in clarifying the relationship between the negotiation of sociomathematical norms and the students' and teacher's cultures. It is important to clarify that we use the term culture in this paper to refer to how individuals participate in these different communities. From this perspective, individuals are viewed as participating in the practices of society, the school and community, and in turn, the classroom microculture. How individuals participate in the negotiation of sociomathematical norms at the classroom is reflexively related to how they participate in the practices of the school and the practices of society. Cast in these terms, individuals are seen as bringing their ways of participating in these different communities into the emerging classroom microculture. In this way, individuals' cultures are related to their mathematical activity through the negotiation of sociomathematical norms.

Data Corpus

The focus of the analysis is to describe how an investigation involving the negotiation of sociomathematical norms may relate individuals' culture to mathematical learning. For this reason, the analysis which follows presents an example analysis of the negotiation of sociomathematical norms for two classrooms. Furthermore, it indicates a direction for more extensive studies examining individuals' culture and mathematical activity. The first classroom was an upper middle class first grade class whose teacher was Caucasian, while the second class was an inner city, predominantly African-American second grade class. The teacher was an African American. Both teachers were

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collaborating with a research team during the time of study and both were trying to promote a math-inquiry style of teaching.

Classroom Analysis

In this section, we present episodes of the negotiation of sociomathematical norms unique to two classrooms. These classroom episodes are used in order to illustrate contrasting situations. We then follow with a discussion comparing the events occurring in each classroom.

Mr. Bill's class

The negotiation of the sociomathematical norm of understanding what counts as a different solution helps students to develop more mathematically sophisticated solution strategies. Mr. Bill guided the negotiation of what constitutes a mathematical difference. His negotiation of this sociomathematical norm relates to how he participates in the practices of the school culture and the wider society. He was attempting to develop a classroom which promotes understanding. Similarly, the students also had their ways of participating in the school culture and the negotiation. It is in this sense that this sociomathematical norm was negotiated between the teacher and students because a student may have offered his different solution not knowing whether the teacher, or other classmates, would see it as different. Hence, both the teacher and students negotiated what it means for a solution to be mathematically different. In the following episode the teacher wrote the horizontal number sentence 22+38+9 on the board. He called on several students who offered different ways to solve the problem.

T: Let's see. how many of you agree with 69? [Students raise their hands]. How many agree with 42? How did you get 69?

Tracy: [comes to the board] I said the 20 plus the 30. That'd be 50. and 50 plus that 2 would be 52. Plus that 8 would be 60 plus that 9 would be 69.

T: Did someone add it a different way? Cathy?

Cathy: I said the 20 plus the 30 equals 50. Plus the 20, I meant 2, equals 52, plus the 8 equals to 60 plus the 9 equals 69.

T: I said different! I did not say what she said. OK. Let's try Jeff.

Jeff: That 8 and that 2 is gonna be 10 and I add it to the 20 (inaudible but probably added the 30 he has with the 30 on the board). and then you have the 9 left.

T: Different? Very Different? Tony?

Tony: Yes.

T: OK

Tony: I said this 2, not this 2. This 2 plus this 8. That was 10. And I said plus this 9. That was 19. and I said this 30 plus 20. That was 50. And I said 19 plus 50 would equal 69.

T: Alright, that's different. different. OK.

Ms. Smith's Class

The ways in which Ms. Smith and her class negotiated what counted as a different solution were very different from the negotiations that occurred in Mr. Bill's class. The ten frame in figure 2 was quickly flashed on the overhead projector and the children were asked to recall how many chips they had seen.

T: Jenny, what did you see?
Jenny: 6
T: How did you see that?
Jenny: I saw three and three.
T: Jenny saw two groups of three. Did anyone else see it the same way Jenny saw it, two groups of threes.
Linda: I think I change my mind to six.
T: OK, How do you see the six, Linda?
Linda Four and two.
T: OK, Linda said that she changed her mind. She sees six but she doesn't see six the same way that Jenny saw six. Jenny said that she saw six as three plus three. So those are two different ways...two different ways to group, to group the pumpkins together and think about six. You can group four together and see four two or you can group the three together and see three plus three...Kayla?
Kayla: I saw it a different way.
T: You have a different way? How did you see it?
Kayla: I saw it as five and one.
T: You did? Kayla said that she saw five and one. [They discuss how many more they would need in order to fill up the ten frame.]
Jenny: I saw...I saw three plus two plus one.
T: OK, that's another way to see six. [Goes on with another problem]

These two classroom episodes provide an example of how the cultural backgrounds of the students and the teachers relate to how sociomathematical norms were negotiated. Both teachers initiated and guided the negotiation according to their pedagogical agendas and their prior experiences in the school culture and society. For example, Mr. Bill indicated to the students that Cathy's and Jeff's explanations were not different from Tracy's. He made a point of emphasizing what counts as a different explanation. Similarly, Ms. Smith did not regard Jenny's second contribution of "3+2+1" as different or more sophisticated than her first explanation of two groups of three. As a result, Ms. Smith did not highlight Jenny's second contribution. It is important to note that the teachers' negotiations are related to their conscious efforts to develop classrooms which promote understanding. Therefore, it was pertinent from the teachers' perspectives to carefully initiate the negotiation of what constitutes a mathematical difference, and in the case of Ms. Smith's class, what is considered a different solution that is more efficient as compared to other strategies.

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Since the ways in which the two teachers guide the negotiation of sociomathematical norms is related to their participation in different communities, we would expect that the teachers would negotiate norms in different ways. The classroom episodes are evidence of this. Mr. Bill emphatically and explicitly indicated that an answer was not different. In response to Cathy, the teacher said, "I said different! I did not say what she said"! This explicitness may be explained by the fact that the teacher regarded this sociomathematical norm as being important. Mr. Bill's explicitness may also be explained by his communication style, which is also related to his prior experiences in the school community and society. Delpit (1988) describes differences in communication styles of African American mothers and white, middle-class teachers. In her description, the communication styles of African American mothers are characterized as explicit as compared to that of middle-class teachers. She describes the communication styles of middle class teachers as being implicit and vague to African American students. It is interesting to note that the negotiation of the sociomathematical norms involves different aspects of individuals' perceptions and communication styles, aspects of their culture.

It is apparent that the negotiations occurring in the two classrooms involved different individual beliefs and perceptions, and for this reason, the negotiation of sociomathematical norms is unique to each classroom microculture. The analysis shows that negotiation of sociomathematical norms may occur very differently in different classrooms. For Mr. Bill's class, negotiation of the sociomathematical norm of what constitutes a different answer was very explicit while for Ms. Smith's class, it was implicit in comparison. Though these negotiations and the involved interactions were different, both classrooms were mathematically productive in that many of the students were able to develop grouping methods and an appreciation for more efficient strategies. Clearly, the different participation of individuals and the different negotiations contributed to an emerging microculture that was unique to each classroom.

The negotiation of social norms also appeared different in the two classes. Further analysis including the negotiation of social norms would be helpful in understanding the classroom microculture and how individuals come to participate in it. Individuals' participation in the negotiation of social norms may also be viewed as being reflexively related to their participation in the practices of the local community and society. In addition, the negotiation of social norms contributes and is strongly related to the negotiation of the sociomathematical norms. Both analyses would provide a more complete picture of the events of the mathematics classroom in regards to individuals' ways of participating in different communities.

Conclusion

I have suggested several key points in regards to examining the negotiation of sociomathematical norms and understanding issues of diversity. Here, I offer a summary of the key points.

1. In viewing sociomathematical norms and the beliefs of individuals as being reflexively-related, the construct of sociomathematical norms is one method of relating the cultural backgrounds of the students and the teacher to mathematical learning. Most of the

literature relating students' and teachers' cultural backgrounds deals with participation an is basically content-free.

2 Clearly, each classroom includes individuals contributing to an emerging microculture unique to that particular classroom. In the analysis, though both emerging classroom microcultures were different and unique, both classrooms were mathematically productive. They were mathematically productive in the sense that students in both classrooms developed ways of grouping numbers and developed understanding of more sophisticated methods. This is a very encouraging finding since each classroom is a uniqu microculture with children of various backgrounds. In addition, this finding has implications for reform, since it suggests that there is not one characteristic classroom tha promotes inquiry math. Classrooms can take on different forms and include different beliefs; however, productive mathematics can still occur.

3 The classroom is situated within the larger context of the school and society. When viewed from this perspective, the events of the classroom are both affected by and affect the school culture and the practices of the wider society. In addition from this perspective, the negotiation of sociomathematical norms may be regarded as events which are related to the school culture and that of the community and society. They are not merely isolated events which are confined to the classroom.

The analysis of the negotiation of sociomathematical norms may indicate a direction for future studies exploring the relationship between students' and teachers' cultures and mathematical activity.

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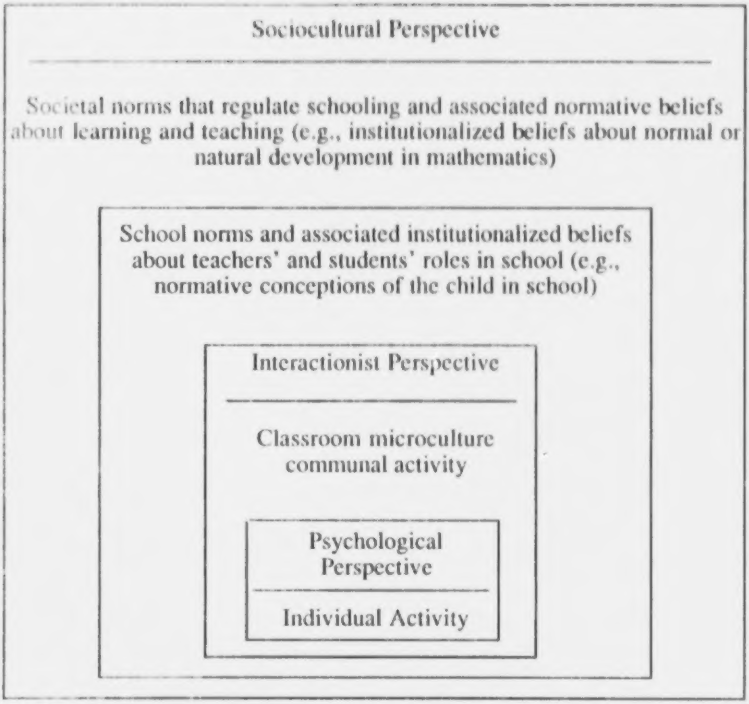
Sociocultural Perspective
Societal norms that regulate schooling and associated normative beliefs about learning and teaching (e.g., institutionalized beliefs about normal or natural development in mathematics)
School norms and associated institutionalized beliefs about teachers' and students' roles in school (e.g., normative conceptions of the child in school)
Interactionist Perspective
Classroom microculture communal activity
Psychological Perspective
Individual Activity

Figure 1 The Elaborated Interpretive Framework

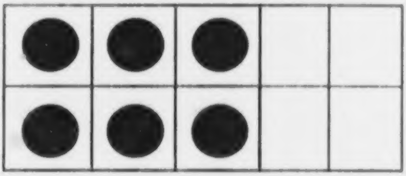

Figure 2 Single ten frame

"AUTOMATISM" IN FINDING A "SOLUTION" AMONG JUNIOR HIGH SCHOOL STUDENTS

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ABSTRACT

This study focused on how junior high school students and teachers find the solution for a given equation, which contains the information, needed to solve it. The aim was to examine whether the participants find the solution through a routine algorithmic process, or whether they employ varied ways of solving the equation. Results indicate that students employ varied strategies, the most common of which is extracting and substituting the variable, while teachers employ two principal strategies: extracting and substituting the variable and direct calculation. Furthermore, the older the participants, the less frequent is their use of the extracting and substituting the variable strategy.

THEORETICAL BACKGROUND

Several researchers who studied the ways of how students arrive at solutions, found that students employ "technical" methods to solve equations. A study conducted in the U.S. on beginning algebra students, showed that students tend to solve algebra problems through a *routine* algorithmic process (Jensen and Wagner, 1981).

In a clinical interview of 8 students, the following problem was given:

Find the value of $\frac{2a+1}{2}=$ when $5(2a+1)=10$ is the given.

The students employed the following patterns of solution:

Method 1: Finding first the value of the unknown

$5(2a+1)=10$

$10a+5=10$

$10a=5$

$a=1/2$

Method 2. Direct calculation (making use of the fact that $2a+1=2$)

$5(2a+1)=10$

$2a+1=2$

$\frac{2a+1}{2}=\frac{2}{2}=1$

Study results indicated that 6 students (75%) used Method 1, one student used Method 2, and one student answered incorrectly. Jensen and Wagner argued that the solution by Method 1 indicates fixation on the routine algorithmic process for solving this type of problems.

Ilany (1991) repeated the experiment in Israel, examining 96 students: 55 at the end of 7th grade and 41 at the end of 8th grade. The results obtained in the study were similar to those obtained in the U.S. However, in addition to the methods that emerged in the Jensen and Wagner study, another solution method emerged.

Method 3: Extracting the expression $2a$

$5(2a+1)=10$

$2a+1=2$

$2a=1$

$$\frac{2a+1}{2}=\frac{1+1}{2}=1$$

Kuchemann (1981) presented a question constructed on the same principle:

If $g+f=8$ then $e+f+g=$

Kuchemann's objective was to examine how students perceive the meaning of letters in mathematics. Namely, do students perceive in this question the given $e+f=8$ as global expression and substitute in the expression $e+f+g=$, i.e., $e+f+g=8+g$, or do they view the letters in a different manner. He found that students ignore the e and f, and solve $e+f+g=$ by relating to g as to something unknown, assigning each time a different meaning to g.

Jensen and Wagner (1981) explained the students' behavior in solving such problems from three theoretical standpoints.

1. Information processing

   Johnson and Wagner explained the phenomenon where children perceive only a single letter as the unknown (the single letter fixation) and are unable to view an expression composed of several members as an unknown, by the fact that students use only the standard procedures which they have learned to solve problems, and are unable to deviate from the familiar. For example, when a student wishes to find the value $\frac{2a+1}{2}=$, he knows from what he has learned that the letter a should be substituted in the expression $\frac{2a+1}{2}=$ (Method 1), in order to obtain the value of the expression. Here the routine process begins, leading him gradually toward the solution. This method was named by Davis, et al. (Davis, Jockush, McKnight, 1978 in Jensen and Wagner 1981) the "Visually Moderated Sequence".

2. Meaningful Learning

   The single letter fixation can be explained by that the student fails to distinguish that it is possible to relate to the expression $2a+1$ just as to x or y, namely an unknown in itself. The student's prior experience restricts him to using a single unknown without expanding his conceptual world to include the "opportunities" of working with more complex numerical expressions.

3. Cognitive Development

   A possible reason for the failure to grasp the concept of variable from the standpoint of the child's cognitive development is the understanding the equal sign. Children tend to perceive the equal sign as an instruction to perform the operation listed to the left of the equal sign and to write the result to the right of the equal sign. They are as yet incapable of understanding that both sides of the equation are equal (Kieran, 1981). Therefore, so explain Jensen and Wagner, when a student is given the equation $5(2a+1)=10$, the equal sign "tells" him that he is to do something with the left side. Thus, the student automatically proceeds with the solution, not noticing the similarity of the structure of the expression $2a+1$ in the equation, and the expression $\frac{2a+1}{2}=$, which is to be calculated.

Following the studies mentioned, we wanted to examine whether similar strategies exist in different age groups and in problems of a different type. The current study focused on the way junior high school students (grades 7, 8, 9) and junior high school teachers find the solution when given an equation which contains the information essential for its solving. The aim of the study was to examine whether the participants find the solution by means of a routine algorithmic process, or use a variety of ways.

THE METHOD

The study focused on 7th grade students (30), 8th grade students (30) and 9th grade students (32), as well as junior high school teachers of grades 7, 8 and 9 (12 teachers). The participants were asked to complete an open questionnaire constructed for the purposes of this study.

The Questionnaire

1. If $3+a=8$ then $3+a+5=$
2. If $3d=6$ then $18d+9=$
3. If $e+f=8$ then $e+f+g=$
4. If $5(2a+1)=10$ then $\frac{2a+1}{2}=$
5. If $x+y=z$ then $x+y+z=$

In each question of the questionnaire the participants were asked to record each stage of the solution and explain.

Question 3 was taken from Kuchemann. (1983), Question 4 was taken from Jensen and Wagner (1981) and Questions 1, 2 and 5 were constructed for the study.

The answers to Questions 1, 2, 4 are numerical, while the answers to Questions 3, 5 are numerical expressions, requiring direct calculation and a global structural perspective.

STUDY RESULTS

Examination of the participants' work produced a variety of different answers. qThe participants' answers were sorted by strategies that were employed, as follows:

Direct calculation - global structural perspective

The strategy is so named because the participants viewed the given and the expression in a global manner. In Question 1, for example, they took the given $3+a=8$ as is and substituted it directly in the expression $3+a+5$, obtaining:

$\underline{3+a}+5=$

$\underline{8}+5=13$

Substitution in the "reverse direction" - global structural perspective,

as in Question 5, for example:

If $x+y=z$ then $x+y+z=$
The solution:
$x+y+\underline{z}=$
$x+y+\underline{x+y}=$
$2x+2y$

Interim method - partial structural perspective (substituting part of the expression), as in Question 4, for example:

If $5(2a+1)=10$ then $\frac{2a+1}{2}=$

The solution by the interim method:
If $5(2a+1)=10$ then $2a=1$ then substitute and obtain

$\frac{2a+1}{2}=$

$\frac{1+1}{2}=1$

Extraction of the variable and its substitution - routine algorithmic perspective, as in Question 1, for example:
If $3+a=8$ then $3+a+5=$
Solution $a=5$, substitute $5+5+3=13$ thus the solution is 13.
Namely, the students extracted the *a* from the given and substituted it in the expression.

Extraction of the variable only - without substitution - partial algorithmic perspective, such as in Question 1, where *a* was extracted from the given $a+3=8$ obtaining $a=5$ without continuing the process further.

Substitution of any numbers, such as in Question 3, for example:
If $e+f=8$ then $e+f+g=$
The participants substituted any number, substituting, for example, $g=2$, $f=5$, $e=3$ and obtained: $3+5+2=10$

Both expressions have the same solution, such as in Question 1:
If $a+3=8$ is given, then the second expression will also receive the value 8, namely, $3+a+5=8$, and therefore the letter *a* obtained a different value in each of the equations. In the equation
$a+3=8 \Rightarrow a=5$
and in the equation $3+a+5=8 \Rightarrow a=0$

In order to compare the strategies employed by the students with the strategies employed by the teachers, we prepared Table No. 1, listing for comparison purposes the teachers' answers in percentage terms, notwithstanding their small number (12).

Table No. 1: Solution Methods - Students Compared with Teachers

Number of Participants	Total Students 92	Total Teachers 12
Extracting and substituting the variable	45.5%	15%
Direct calculation - structural perspective	29%	85%
Interim method - substituting part of expression	1%	0%
Substituting any numbers	14%	0%
Extracting variable only - no substitution	1%	0%
"Reverse" substitution	0.5%	0%
Same solution for both expressions	0.5%	0%
No answer	8%	0%

Table 1 shows that the teachers employed only two strategies: extracting and substituting the variable and direct calculation. The students, on the other hand, employed other strategies as well.

The most common strategy used by students of grades 7, 8 and 9 was extracting and substituting the variable (45.5%), while teachers most commonly used direct calculation (85%).

Table No. 2: Results Obtained - By Grades

Number of Participants	Grade 7 30	Grade 8 30	Grade 9 32	Teachers 12
Extracting and substituting the variable	43%	45%	46%	15%
Direct calculation - structural perspective	15%	31%	41%	85%
Interim method - substituting part of expression	0%	1%	3%	0%
Substituting any numbers	28%	12%	4%	0%
Extracting variable only - no substitution	2%	1%	1%	0%
"Reverse" substitution	0%	2%	0%	0%
Same solution for both expressions	1%	2%	0%	0%
No answer	11%	6%	5%	0%

In Table 2 we can discern the use of three principal methods: extracting and substituting the variable, direct calculation (methods used by both students and teachers) and substitution of any numbers (used by students only).

In the extraction of the variable strategy, no difference in use was discerned between grades 7, 8 and 9. On the other hand, in the direct calculation strategy, there was a discernible difference between the grades: the higher the age, the more frequent the use of this strategy (in grade 7 - 15%, in grade 8 - 31%, in grade 9 - 41% and teachers - 85%). Moreover, use of the strategy of substituting any numbers, an incorrect strategy, decreases with increasing age (in grade 7 - 28%, in grade 8 - 12%, in grade 9 - 12% and teachers - 0%).

DISCUSSION AND CONCLUSIONS

The **variable extraction and substitution** strategy, commonly used by students, is used less frequently with increasing age. This can be explained by the perception of using a letter for a specific number only. Many students believe that variables are letters representing numbers only (Usiskin, 1988, Ilany, 1996).

Such use of the **variable extraction and substitution** strategy was also found in the studies of Jensen and Wagner (Jensen and Wagner, 1981), who explained it from three standpoints: Information processing, meaningful learning and cognitive development (as detailed in the theoretical background).

In relation to information processing, we found, just as Jensen and Wagner, that:

1. Some students solve problems by means of a familiar standard procedure which they learned and from which they are unable to deviate. For example: in Question 2 where the given is $3d=6$, students first find the value of d and substitute in the expression $18d+9$.
2. Some students are not proficient in other solution methods save that of finding the unknown and its substitution, and therefore solve the problem by the method familiar to them.
3. When a student wishes to find the value $\frac{2a+1}{2}$ he knows from what he has learned, that a is to be substituted in the expression $\frac{2a+1}{2}$, in order to obtain the expression's value. Here begins a routine process which gradually leads the student to the solution.

In relation to meaningful learning, it is possible to explain the single letter fixation by that the student does not realize that expression 3+a can be treated just as x or y, i.e., an unknown in itself. From here it may be inferred that the student has not as yet fully formed and internalized the concept of variable. This finding testifies to the student's limitation in understanding the concept of variable. The experience to which the student has been exposed, restricts him to using a single variable and does not expand his conceptual world to perceive the possibility of working with more complex numerical expressions.

From the standpoint of cognitive development, the use of a letter as a specific number can apparently be explained by the variable being perceived, at the beginning of algebra, as a number only.

The teachers' most frequent use (85%) of **direct calculation**, can be explained by the concept of algebra as study of structures. In other words, learning algebra at the tertiary level, according to Usiskin (1988), is usually characterized by dealing with structures, a perception in which the variable is viewed as a random object within an expression, related to other variables in the expression by certain properties. This approach is characteristic of abstract algebra, and we thus see in this study that teachers possess the ability to deal with variables without requiring a numerical frame of reference. Moreover, teachers have been exposed to a broader variety of uses of variables, and have expanded and deepened their understanding of the concept of variable. Our study shows that with higher chronological age, more participants shift to direct calculation. Namely, with increasing age, the concept of variable expands, learning becomes more meaningful, and more participants view 2a+1 or 3+a as an unknown in itself.

The use of the **substitution of any numbers** strategy (28/% of 7th graders, 12% of 8th graders, 4% of 9th graders), can be explained in several ways:

1. The child perceives the meaning of the variable from the numerical aspect.
   Only numbers can be substituted for letters and they therefore substitute any numbers for letters.
2. The "incomplete nature" of several expressions - research findings show that students tend to "simplify" expressions such as 2a+5 to 7a or to 7. This tendency can be characterized by the need to provide a "final" numerical answer (Tirosh, Even and Robinson, 1994).
3. The "process is also the result" -
   One of the difficulties encountered by students in the transition to algebra, is connected with the need to relate to an expression as representing concurrently both a process and a result (Davis, 1975, Booth, 1988). The source of this difficulty stems from the habit of separating the two in arithmetic. For example, in 8+5=13, 8+5 is the process and 13 is the result.

It was interesting to find in our study the **interim method - substituting part of the expression**, as in Question 4, for example:

If $5(2a+1)=10$ then $\frac{2a+1}{2}=$

The solution by the interim method:

If $5(2a+1)=10$ then $2a=1$ then substitute to obtain: $\frac{1+1}{2}=1$

This method was also found in solving Questions 3 and 5, as in Question 5, for example:

If $x+y=z$ then $x+y+z=$

The solution by the interim method:

If $x+y=z$ then $x=z-y$, substitute for x in the expression $x+y+z=$
and obtain $z-y+y+z=2z$.

In Question 3: If $e+f=8$ then $e+f+g=$

The solution by the interim method:
If $e+f=8$ then $e=8-f$, substitute for e in the expression
$\underline{e}+f+g=$ and obtain

$\underline{8-f}+f+g=8+g$

The **interim method - substituting part of the expression** strategy was also found in Ilany (1991)

We were thus surprised to find in our study, that although the teachers themselves employ direct calculation, many students do not use this method. Perhaps some improvement could be obtained by exposing junior high school students to the various uses of the variable and varied solution strategies, leading the students to the desired structural - global perception.

To summarize, a long road is traveled by the student in the process of learning algebra: at the outset, the letters are perceived as representing numbers only, and further on, a more global structural perception is attained.

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THE ROLE OF CONTEXT IN COLLABORATION IN MATHEMATICS

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This study explored the role of context in students' construction of mathematical concepts. The discourse of 8 pairs of students, aged 11 and 12, was recorded while they solved mathematical problems together. In each pair a more able peer worked with a less able partner on either contextualised problems or purely numerical problems. Those students who made the most progress were those who worked on contextualised problems. Analysis of the discourse demonstrates that it was the context that provided clear goals and drew on the experience of both partners, so that the more able were able to learn from the less able as well as vice versa.

Analysis of the discourse in mathematics education is a fertile source of information on what is enabling learning. One example was the examination of discourse in reform classrooms, showing the emergence of a collective voice in a classroom (Graves & Zach, 1997). Less attention has been paid to what the students are talking about, and the effect of context on learning from this discourse. In peer collaboration, we need to evaluate this discourse and who benefits from it.

Added depth is given to this analysis through looking at it in relation to the different views of Vygotsky and Piaget on learning from peers. Vygotsky (1978) believed that more capable peers, or adults, were needed for learning because they understood the goal of the activity and knew what it was that the learner needed to know next to gain that goal. Piaget (1932/1965) postulated that children were more likely to reflect on and change their own concepts as a result of discussion with a similar peer than with an adult. He gave the conditions necessary for learning from a peer as having common language and intellectual values, being able to conserve one's own ideas while working for consensus, and being able to consider each other's propositions in a reciprocal manner. This paper argues that context can be a crucial variable in producing the characteristics described by both of these theorists.

Other theories of learning acknowledge that context is essential in learning. Apprenticeship theory, for example, places emphasis on the entire context in which a set of practical or cognitive concepts are learned. In apprenticeship learning the goal of learning is clear and the steps along the way to meeting this goal can usually be articulated in a manner that enables the more able partners to help the less able partners meet these goals. Learners know that a solution that meets obvious criteria is to be found. It has been proposed that the clearer such goals are to the learners the more likely they are to meet them (Resnick & Resnick, 1993).

In New Zealand the primary context for learning mathematics is the schoolroom itself. It is the context for patterns of discourse, scripts, goals, and patterns of respect (see Leinhardt & Putnam, 1987). This is such a major feature of context for students that they often forget everyday contexts in which they have learned certain things. Some students are skilled in learning mathematics in a school context. They understand and follow teachers' instructions, grasping many facts about the manipulation of numbers. In the upper elementary school they understand that the goal of mathematics is to deal with numbers in a formalised manner, with numbers being treated as entities without further context. They often value this formalisation to the extent that they ignore contexts that give these numbers different meanings. Other students are less skilled in learning in classrooms. The school context, although familiar to them, is less meaningful. They follow the classroom scripts and activities but may not understand the mathematics in these activities. They may come to think of themselves as less capable in school mathematics and having less to offer their peers in collaborative sessions. Out of school, on the other hand, they are confident in many spheres in which numbers are used. Walkerdine demonstrated that the relationship between out-of-school experiences and classroom mathematics is complex, especially for poorer students such as those in this study (Walkerdine 1997).

This study was carried out in a school, albeit in a withdrawal room. One expectation of working on problems in such a setting might be that both partners would utilise the beliefs and activities that are valued in classrooms. The less competent peers might defer to the greater formal knowledge of their more competent peers as they usually do in classrooms. There were many cases in which this did not happen. When problems were presented in a familiar context, the less competent peers used their out-of-school knowledge rather than their formal knowledge, and challenged the formal knowledge of their more competent partners. Because both partners were familiar with this everyday knowledge, the more competent peers listened to the challenges presented by their partners from this everyday knowledge. Between them they recognised the conflicts and worked to combine their expectations from these context with their formal knowledge to reach a conclusion that met an understood goal.

The Study. This study was the third of a series designed to examine the effect of context on learning about decimal fractions. It followed two studies that explored the contexts in which 84 similar students used decimal fractions out of school, and the errors that they commonly made (see Irwin 1995).

Sixteen students, from a lower economic, ethnically mixed class took part in this study. The students were ranked by their teacher on their overall mathematical ability, and were then placed in pairs with a constant difference in ranking between them (1 & 9, 2 & 10, etc.). Thus they were paired to meet Vygotsky's criteria, yet the behaviour of some of the pairs matched Piaget's criteria.

Mathematical problems were developed that were based both on contexts that the prior studies had shown were familiar, and on errors that were commonly made by the students interviewed. A similar set of problems were also written which used formalised word problems without everyday contexts. Half of the pairs worked on contextualised problems and half worked on similar problems in a purely numerical form. Examples of these two types of problems were:

> *Contextualised*
> How much do you think you will have left if you have a 1.5 litre bottle of drink and you pour out enough to fill a 225 ml glass? (bottle and glass on display)
>
> *Numbers only*
> If you subtract 0.225 from 1.5 what will you get?

All students were given a pretest of ordering decimals and operating with them in a formalised manner. They then worked together on four problems per day for the next three days. These problems covered magnitude, addition and subtraction, and multiplication and division of numbers that included decimal fractions. Questions were designed to draw attention to common confusions. Students were given a posttest, very similar to the pretest, four days after their last problem-solving session and again one month later. The discussion between the partners was tape recorded and analysed at a later date.

Analyses. Students who worked on contextualised problems made significantly greater gains between pretests and posttest than did those who worked on purely numerical problems ($F(1,12)=5.70$, $p=.03$). These students were also significantly more accurate in solving the problems done jointly ($F(1,12)=6.21$, $p=.02$).

Working on contextualized problems helped both the higher ranked and lower-ranked members in the pair to increase their posttest scores.

The analysis of the discourse explored what might support this improvement. The statements made by each partner were categorised by the contribution that they made to the conversation. Categories were based on the argument operations used by Azmitia (1988) and Pontecorvo and Giradet (1993). The categories used were:

- Answer (right or wrong)
- Explanation (right or wrong)
- Agreement
- Disagreement
- Challenge
- Question
- Consideration of other's view
- Request for agreement
- Comment
- Incomplete statement

The hypothesis behind this analysis was that some types of argument were more likely to advance understanding than others. Students who listened to one another learned more than did those who worked in parallel (see Forman, 1981; Ellis, 1995). The balance of the partners' statements would indicate if both offered answers and explanations and challenged one another, or if one of the partners made most of the statements that led to solution of the problem.

There were marked differences in reciprocity when students were working on contextualised or purely numerical problems. Two excerpts are given below to demonstrate these differences. The first is from a pair working on purely numerical problems who used the scripts appropriate for formalisms used in a school context, but which they did not fully understand. The next is from a pair who worked on contextualised problems. For them the context provided the chance for the less able student to provide a lead to the partner who was more skilled in formal mathematics.

Pair 3 who worked on purely numerical problems was made up of two girls, aged 11. Neither partner made significant gains. Their relationship was friendly but the lower-ranked partner made few substantive contributions, preferring to defer to her more capable partner. There was a marked imbalance in the types of statements made by each of the girls. On this problem the more capable partner, H, gave 37 partial or full explanatory statements and the less capable partner, A, gave 6 explanatory statements. H agreed 4 times and A agreed 20 times. Both partners asked each other questions but there was little genuine conversation. This pair lacked the common intellectual values and reciprocity that Piaget notes as essential for collaboration. A did not appear to work for the conservation of her ideas in the face of H's statements. The written question is given in italics and has two parts.

(1Be)*Teri said that* $93\frac{1}{4}$ *was written as 93.04 in decimals. Why did she say that? Do you agree?*
A Yes - Oh \ WRONG ANSWER
H Yeah, 93 is a whole number and one quarter is just, it should be zero zero four - I think. AGREEMENT, DIFFERENT WRONG ANSWER
A No. Try that again. 9 3 point \ CHALLENGE
H Oh remember, remember, cus decimals leave, you know how this goes ones, tens hundreds, this goes hundreds, tens, ones. (points to place-value columns for hundreds, tens, ones, oneths, tenths, hundredths) EXPLANATION
A Oh yeah, and then \ AGREEMENT, INCOMPLETE
H But you see that can - umm. Do we agree? Cus the four should be, there should be another zero between the zero and the four. REQUEST FOR AGREEMENT, WRONG ANSWER
A Yeah. AGREEMENT
...

Peta said that $93\frac{1}{4}$ *was written as 93 .25 in decimals. Why did she say that? Do you agree? What do you think is right?*
H (23 second pause) Hold on. (pause) Yeah, it could be right . CONSIDERS OTHER VIEW, TENTATIVE AGREEMENT
A (starts to say something, interrupted) INCOMPLETE
H See most, that is one quarter. 25 is one quarter \ EXPLANATION
A Yeah. AGREEMENT
H In decimals. EXPLANATION
A Yeah, so \ AGREEMENT
H Do we agree? REQUEST FOR AGREEMENT
A Yes. AGREEMENT
H Yup we agree. AGREEMENT
... (35 lines later)
A This one (93.04) could be right though. (9 second pause) CHALLENGE, BACK TO WRONG ANSWER
... (27 lines later)
H It could be both. Could it be both? Or just one of them? (pause) TENTATIVE ANSWER
... (problem eventually solved by H)

In contrast, Pair 1, who worked on contextualised problems, shared the task relatively equally, and both gained in knowledge. The more advanced student, M, was a girl, aged 12 and the less advanced, N, was a boy aged 11. They listened to each other and finished each other's sentences. Both partners contributed, and both corrected each other. Both partners were wrong on occasion and challenged by the other, and both appeared to benefit from this challenge. In their discussion about problem 2Ac, part of which is given below, their dialogue was balanced. Each partner gave 7 answers; M gave 10 explanations and N gave 8; Each disagreed or challenged their partner 5 times; and M considered her partner's view 2 times while N did so 3 times. The more advanced partner initially treated the problem in a formalised manner, lining up the decimal points and adding. The less advanced student used his everyday knowledge to challenge her. She quickly acknowledged the correctness of his response and rethought her answer. It will be seen later that she challenged his answer of 6.05.9 because it did not fit the rules of formalisation, despite the fact that it did make sense in an everyday context.

2Ac *If you go on a trip and you buy 1 litre of petrol at 90.9c and a meal at McDonald's at $4.95, how much will it cost?*
M Ninety five dollars about, something cents... four cents. WRONG ANSWER
N Or five dollars.. cus its the whole thing. DIFFERENT ANSWER
M Yeah I know, ninety five dollars and something cents Probably about ninety five dollars and four cents. AGREEMENT (not listening to partner)
M Where's the petrol? CHALLENGE
M That's how it costs all together isn't it? QUESTION
N Ninety five dollars?... for a McDonald's and petrol? CHALLENGE
M Yeah (laugh) oh. CONSIDERATION OF OTHER'S VIEW, AGREEMENT
...
N Five dollars, six dollars... six dollars point eight, no point, no six dollars five.. uh six dollars five cents point nine. ANSWER
M Six dollars five cents point nine? CHALLENGE
N Yeah.
M Point nine? (Pause) six dollars and five cents point nine. CHALLENGE
N Yeah.
M That means six point five point nine. EXPLANATION
N Oh man nah... CONSIDERATION OF OTHER'S VIEW, AGREEMENT
...

These students solved the problem by changing 90.9 c to 99c (M's solution) because they knew how to operate with that. They then added the numbers without the decimal point, translating $4.95 to 495 cents (N's solution). This enabled them to get an answer that was in line with their goal which was a sensible estimate based on context.

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Discussion. All of the dialogues showed some of the characteristics of a discussion in which school contexts and formalisms were uppermost in the peer collaboration. However, it was the use of context, as in the second example, that enabled the partners to learn.

The first dialogue presented depended entirely on formal use of numbers. Explanations related to recalled classroom scripts, including the place whose value is "oneths". Extra zeros are added without an understanding of what this does to the value of the numeral. The dialogue was unbalanced with the less knowledgeable saying less and deferring to the authority of her more capable partner, whether or not the mathematics made sense to her. Their goals were different. From the full transcript it was clear that H's goal was to recall or deduce the relevant knowledge and A's goal was to agree to an answer so that they could move to the next problem. The second dialogue was marked by greater equality of contribution, with the lower ranked peer offering the everyday knowledge ignored by the higher-ranked peer who attempted to use the formalised knowledge valued in the classroom. The higher-ranked peer accepted the value of the everyday knowledge of the lower-ranked because those contexts also made sense to them. This everyday knowledge acted as a shared goal. Their appropriate estimates of what the answer would be helped them see the steps to that goal.

All of the discussions took place between a more capable and a less capable partner, yet the dialogue of the first pair does not meet either Vygotsky's or Piaget's criteria for learning from peers. The second pair would meet Piaget's description of equals with different perspectives who respected one another's propositions and sought consensus, while also meeting Vygotsky's description of a more and less able peer working toward known goals.

Problems carefully constructed from familiar contexts did enable students from a poorer area to bring their knowledge into school to aid their learning of mathematics. The problems did enable them to see that they had knowledge to offer that was more valuable than the formalisms that they usually valued in school. However, these contexts had to be selected with care. One of the most useful in this study turned out to be students' knowledge of a 1.5 litre bottle of soft drink and the fact that it was roughly equivalent to six glasses. This group were also familiar with exchange rates as many of the students from Pacific Countries had either travelled or seen relatives send money overseas. These contexts might not be appropriate for other groups. The problems also needed to be developed to challenge students' misconceptions. It was the everyday contexts that provided the goals for students and encouraged them to check their answers, letting them see that, even in decimals, answers can make sense.

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Design and Evaluation on Teaching Unit :
Focusing on the Process of Generalization

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Abstract

"Mathematization" is a terminology to extend the educational philosophy of mathematical teaching and learning as an active and creative process. It is Wittmann's Teaching Units that has brought the idea of design science into the didactical practice of mathematization. His Teaching Units, however, does not have a diachronic design of how the lesson in classroom should be developed over time although it contains the fundamental design such as objective, material, problem, and background. On the other hand, the mathematization can be regarded as a typical symbolization process in the sense of Peirce which is called semiosis, and in this context the Dörfler's generalization model, which starts from the activity and constructs the generalization process, is quite thought–provoking.

Therefore, it is possible to design, practise and evaluate the Teaching Units of Wittmann in the teaching–learning context based on the generalization model by Dörfler. This is the objective of this research and at the same time the result which we could get. The methodology is to have employed a teaching experiment 'Star Patterns' based on the above theoretical framework and to prove its effectiveness through the analysis of results in terms of Dörfler's model.

1. Objective and Methodology

(1) Objective

Freudenthal, as a mathematics researcher, has a didactical belief as follows "There is no mathematics without mathematizing. ···This means teaching or even learning mathematics as mathematization. ··· This is what follows from the interpretation of mathematics as activity."(1973,p.134)

He theoretically prepared a didactical foundation of mathematization as an Educational Task and Phenomenology, but he has not presented a concrete example systematically together with a clear framework and ideas on the didactical practice of mathematization. In that sense, Wittmann has brought about an idea of design science into it, given the framework of Teaching Units(hereafter cited as TU), and realized many examples of TU for mathematization(Muller,G and Wittmann,E. 1977)

The objective of this research in the following two points. The first point is to clarify the theory and methodology of TU by Wittmann and to point out some problems involved in this theory. The second one is to solve these problems, to put this TU into the classroom practices, and to evaluate this practice through introducing the generalization model of Dorfler. Briefly

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speaking the characteristics of this research is to complement TU by Wittmann with Dorfler's theory.

(2) Methodology

We have analysed the classroom lesson mentioned below in section 3 as we always examine whether the record can be a teaching experiment. But we are quite encouraged by the enthusiastic discussion between two researchers in Working Group 25 "Didactics of Mathematics as a Scientific Discipline" of ICME-8 in Seville Spain 1996. Hans-Georg Steiner stressed that the theoretical framework should be given priority over the 'fact'. On the other hand, Michele Pellerry who is a Italian psychologist emphasized his stance that the framework by itself cannot come first and mathematics education should be a scientific discipline built on facts, experiments and inductivism.

Which standpoint and viewpoint are right could not be determined by experiment like as the case of the theory. We, therefore, would like to withhold ourselves from decision whether our record on classroom lessons is a teaching experiment. But we must say that we planed them deliberately, and recorded them by video carefully, and investigated them by video together with the concerned teacher and other researchers many times because the classroom lesson cannot be done again.

Steiner asserted that the situation could not become the phenomenon without the theoretical framework. In other words, the 'fact' of Pellerry can be defined clearly only under the theoretical perspective. Even the case study done by constructivists has the same tone as that of Steiner in our understanding. We proudly think that we could change the situation into the didactical phenomenon of mathematics through our framework.

We are always wondering whether the well controlled teaching experiment is consistent with being educational in didactical practice. The strict conditions on science remind of the European proverb. Don't we throw away the baby with bathwater. Our conclusion in this case study can be summed up into a viewpoint on the design of classroom lesson and a theoretical framework for the diachronic assessment on it. But we believe that teachers can reach to the similar considerations on our teaching practice if they understand and are interested in the viewpoint and the framework presented here.

2. Teaching Units

(1) Wittmann's Didactical Philosophy before Introduction of TU and its Critical Consideration

Wittmann conveyed his views towards mathematics and school mathematics to support his mathematics education in the process of identifying a core research in mathematics education.

> "Mathematics teaching is doing mathematics with students in order to cultivate their understanding of reality."(1984,p.29)
>
> "Mathematics educators must be aware that school mathematics cannot be derived from specialized mathematics by a "transposition didactique du savior savant au savior enseigne" (cf., Freudenthal 1986). Instead, they must see school mathematics as an extension of pre-mathematical human capabilities which develop within the broader societal context provided by MATHEMATICS."(1995, p.359)

These quotations reflect his view of the internal mathematics which comes from human autonomic activities. It doesn't seem to him that mathematics is an external system of objective knowledge for transmission. At the same time, it is pointed out that internal mathematics emerges through the heteronomic framework of social interaction. Thus the important didactical issue for Wittmann is how to balance autonomic and heteronomic based on what criterion.

It is nonsense to seek a solution within the existing system of knowledge for the problem of creation, cognition and transmission of knowledge. The solution, therefore, should be sought in MATHEMATICS as a situation with its implication. His basic idea is that the role of didactics of mathematics is how to translate appropriately MATHEMATICS as a situation into classroom teaching, and we believe that his proposal is TU itself.

However, as mentioned in (2), the design of TU has objective , material, problem and background, but doesn't contain the design for the classroom practice as a diachronic process of development. This is why the didactical significance has not been described enough for the teaching-learning process of TU. In other words, TU, in nature, doesn't have an analytical system for how the internal mathematics of children resonates with MATHEMATICS as a situation in the social context and how they are shared among them as a knowledge.

We regard language and sign as a symbol of knowledge creation and also as a necessary device for thought and transmission of ideas. That is to say both autonomic cognitive process and the heteronomic social interaction in mathematics teaching and learning are regarded as a process of symbolization. In our opinion, the mechanism of cognitive development in mathematics can be clarified in terms of Dorfler's "symbols as objects". Therefore, our conclusion is that the diachronic development of any TU into classroom can be to some extent designed and evaluated according to Dorfler's generalization model.

(2) The Reinforcement of Wittmann's TU by Dörfler's Generalization Model

Many TUs are always designed in terms of four elements(Wittmann,1984, pp.30-32, 1995, pp.365-366): *Objectives* of TU, *Materials* of TU, mathematical *Problems* arising from the context of TU, and mathematical *Background* of TU. They could compose a TU as if they were four pillars for the building of TU. Both the beginning and end of TU could be clarified by taking them into consideration, but they cannot compose a continuous teaching process as the diachronic phenomenon. In other words, these are not to describe and evaluate in detail the teaching practice based upon TU. Therefore, it is hard to think that the theoretical framework for placement of TU into the process of teaching has been prepared beforehand.

As mentioned in (1), the conceptualization of knowledge via symbolization is the essence of the teaching-learning process in mathematics education. Of course, this conceptualization should involve the active participation by children. Therefore, it is difficult to analyze diachronically mathematics learning unless we assume the symbolization process of internal relations embedded in the activity and the cognitive process which supports this symbolization.

It is Dorfler's generalization model(1991) as shown in Fig. 1, which regards generalization as the essence of mathematical conceptualization and describes the generalization process starting from activity from the perspective of symbolization. The pre-stage of "symbols as objects" is called "constructive abstraction", the objective and method of activity are reflected upon first and the properties of elements of activities and the relation of elements are extracted as a result. Furthermore, the applicability of these properties and relation is considered, "extensional generalization" After this stage, symbolical description of the invariants is removed from the original context and the symbol itself has become object of thought This is to say "symbols as objects" The symbol here is an object as a variable and its connotation has general structure And this structure supports "intensional generalization"

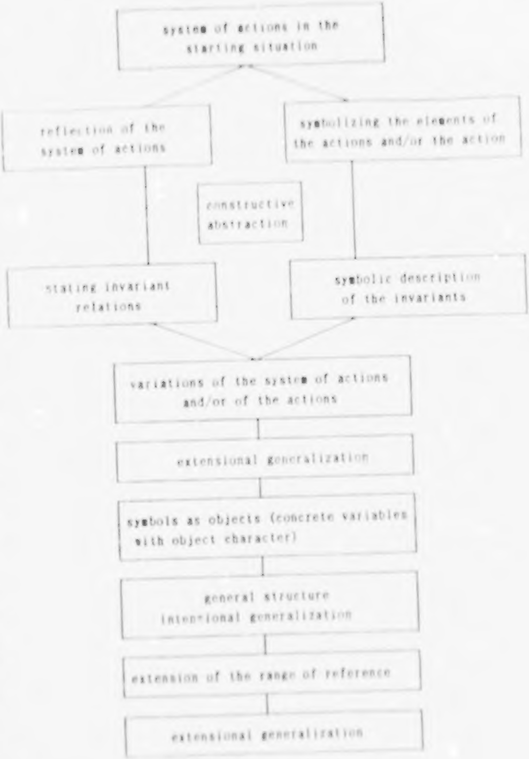

system of actions in the starting situation
reflection of the system of actions
symbolizing the elements of the actions and/or the action
constructive abstraction
stating invariant relations
symbolic description of the invariants
variations of the system of actions and/or of the actions
extensional generalization
symbols as objects (concrete variables with object character)
general structure intensional generalization
extension of the range of reference
extensional generalization

Fig.1 Dorfler's generalization model (1991,p.74)

The symbol as a variable behaves independently as an object of thought in the "symbols as objects", which is the end point of "constructive abstraction", and the internal thought of the individual can be socialized to the external thought common to members of classroom through the symbols as objects Thus this model of generalization gets uniqueness by placing the symbol in the model, noting the role of symbol as a medium, and this offers many implications for mathematics education from the didactical viewpoint This means the analytical device by means of students' discourse and description can be found here for the cognitive process which propels generalization in "symbols as objects" as a milestone However the cognitive system to analyze it should be prepared in advance and 'expanded meta-cognition' (Iwasaki and Yamaguchi, 1997) is available for this purpose Hence the Dorfler's generalization model supplemented by 'expanded meta-cognition' reinforces the idea of TU, i e it enables us to design, practise, and evaluate the teaching process of TU based on the above four elements

3 Design and Practice of TU "Star Patterns: What can you get by connecting points? "

(1) Design of this TU

The objective of TU "Star Patterns"(Bennett,1978, Hirsch,1980) is to find a rule and generalize it between the figures formed by connecting points equally spaced along the circumference, especially some stars, and the way those points are connected. The basic design of this TU is as shown below, and the main purpose of this teaching experiment is to practise and evaluate this TU based upon the Dorfler's generalization model.

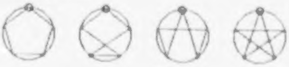
The Design of TU " Star Patterns"

Objective: to extract some properties of stars and to generalize the properties through symbolization

Material : Star patterns

Problem: ① What figure is formed by connecting two, three or four equally spaced points along the circumference in one stroke? The figure formed *in one stroke* means to be composed of only one closed line.

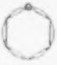
② Connect five equally spaced points along the circumference in one stroke and name the formed figures.



③ Take the star figure among formed figures in the above into consideration. What is the way of connection when this figure is formed?

④ Draw other star figures by connecting six, seven or more points equally spaced along the circumference in one stroke.

6-1



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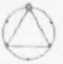

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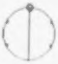

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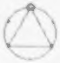

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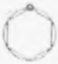
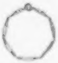


7-1



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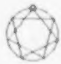

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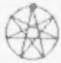

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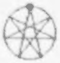

(The rule is to connect points by line and in one stroke. Start with the marked point and connect points every d points. Decide how to name the figure to avoid confusion.)

⑤ When can the star figure be formed or not formed? And what is the reason for this?

Background: When connecting n equally spaced points on a circle every d points in one stroke, the following relation holds.

- The regular star polygon of type n-d is defined for n equally spaced points on a circle if and only if the formed figure satisfies the following three conditions, ① to connect all n points, ② to connect them every d points, and ③ to connect them in one stroke. Therefore, following this definition, the regular star polygon includes not only n-sided nonsimple polygons like 7-2, 7-3 and 7-4 but also regular convex n-gon like as 6-1,6-5 and 7-1 in the regular star polygons.
- If n and d are relatively prime, it is possible to draw a regular star polygon by connecting all points along circumference in one stroke.

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- If n is a multiple of d, then the formed figure is the regular convex n/d-gon.
- If G.C.D. of n and d is g, which is not equal to one , then the formed figure is the same as star polygon formed by connecting n/g equally spaced points every d/g points.

The correspondence between the teaching experiment with this TU and Dorfler's generalization model is as follows. In the problems ① and ② the students are supposed to connect in one stroke some equally spaced points along the circumference. In the problem ③, focusing on the case of star figure among the figures formed by connecting five equally distanced points, the students are supposed to describe determinant activity, 'connecting every two points'. In the next problem ④, the students are required to examine whether the property found in ③ holds with other star figures. This is "extensional generalization" in the first stage. In the problem ⑤, the establishment of a mathematical relation as the objective of this TU is attained and it is sought to share that relation among the students in the classroom.

For example, in problem ④, the students may symbolize the figure formed by connecting six equally spaced points every two points as 6·2, but it becomes practically as an object of thought afterwards. Further, it is important to change it to variable by means of such representation of letters as 'n equally spaced' and 'every d points'. In short, this is "symbols as objects" in the Fig.1. Therefore in the problems ④ and ⑤, the critical point for evaluation of "symbols as objects" is the transient stage from number to letter in the usage of symbol. This is why it becomes possible to grasp from the student's discourse and their used symbol the progressive situation in the generalization stage.

(2) Classroom Teaching and its Evaluation

We conducted a teaching experiment of three lessons according to the teaching design with grade 8 students in Koi Junior High School, Hiroshima. The class was divided into six groups of 6 to 7 members so that they felt it easy to make a discussion.

The students, after understanding the problem context and the method of connecting points according to the rule, are enthusiastically engaged in this activity. Especially they had fun with connecting the points equally spaced along the circumference and made steady progress in that activity through confirmation of figures among the members of group. Continuing with this activity, they have some identical figures formed and are induced naturally to explore the relation among them.

The relation found by the students starts with simply 'the same as' at the beginning and they proceed to the analysis of properties for symbols, that is a set of numbers in which the figures are identical. The relations to find are such as "A segment is formed when n is just twice d.", "A regular convex n-gon is formed when n is divisible by d.". "The figure formed is the same as the figure for the set of numbers which are n divided by G.C.D. and d by G.C.D. if G.C.D. of n and d is not 1.", and "A regular star polygon is formed when n and d are irreducible." The predicates of their statements and discourse gradually get precise and correct mathematically. We can observe clearly that the discovery by an individual student stimulates the discussion from simple figure to more complicated one and from reducible numbers to irreducible ones.

The relations which students have written on work sheets are classified from the viewpoint of symbols as follows

a) The same figure can be obtained for 6·2, 6·4, and 9·3

b) The same figure can be obtained for 6·2 and 9·3 In both cases the quotient is 3 and the equilateral triangle is formed

c) An equilateral triangle is formed when the quotient is 3 *such as* 6·2 and 9·3

d) A regular convex n-gon is formed with ☐ vertexes for ☐ ·1

e) A regular star polygon is formed when x is not divisible by y in x·y

The description a) is about the congruent figures The symbols, 6·2, 6·4, and 9·3, are names given to each figure The descriptions from b) to e) are relations to be found when figures and numbers are interlinked and pairs of numbers such as 6·2 and 9·3 are an object of consideration here However, while in b) only the specific figure drawn on paper is dealt with, in c) 6·2 and 9·3 are mere examples and it is shown that this relation holds with other pairs of numbers This is characterized in the expression, 'such as'. The expression about the relation expands its denotation through acquisition of generality but "symbols as objects" is not attained yet The description d) may be a transient stage from the specific number to the letter as a variable In e) the total number of equally spaced points along the circumference and the distance of connected points are made variables in terms of x and y This shows "symbols as objects" can be achieved

The description and discourse by students in a) to e) can be regarded as external expression of internal thought The expressions a) to e) can be analyzed from the perspective of cognitive theory as follows The expression a) can be obtained by focusing on the visual similarities between figures and, on the other hand, the expressions from b) to e) can be obtained by abstraction of the relation between drawn figures and their numbers And unless due attention is paid to such a view, the transition from the stage a) to the other stage cannot be fulfilled Meanwhile, the gap between b) and c) depends upon whether the consideration can be directed towards extensive generalization which is implied by the word "such as" and the sentence form "if ~ , then". The consideration in the case c) can be called meta-cognitive because it reflects the applicability of the hypothetical schema concerning drawn star polygons, and it determines the difference between the descriptions b) and c) whether the invariants are considered In any case, the considerations of b) and c) are confined to the inductive inference based upon the specific figures in the problems, and the relation obtained from observation of those figures is hypothetically formulated as personal schema

Henceforth the descriptions in b) and c) are not necessary to explain or persuade the others, because of its visual self-evidence, but they are quite defenseless to the question 'Why?' out of other students or teacher The deductive inference is necessary to surmount the limitation of perceptive explanation and it can be achieved together with symbolization and variablization of the relation As being considered in the above, the descriptions in d) and e) can be regarded the transient stage towards symbolization of relation or the stage in which that symbolization is attained, but the objective of the consideration behind them of course differs from the one of the previous stage This is to say the objective of consideration here is to verify deductively the personal schema, i e a hypothetical description of various relations underlying star polygons The the descriptions in d) and e) could not be reached if it were not for meta-cognition which regarded hypothetical schema as an object of considerations and conceptualize it And the source for such meta-cognitive consideration can be sought in the social interaction in the classroom It is TU based on MATHEMATICS that provides the appropriate situation for the social interaction

While evaluating the activity by student, the value of this TU as a teaching material should be evaluated simultaneously In fact star polygons have more mathematical properties than listed here It is possible for example to extend the definition of regular convex n-gon in terms of regular star polygon with n/d vertexes, and furthermore to deduce a group theory from the point of symmetry In this sense this is a teaching material for TU, which has a lot of potentiality and appropriate material for experiencing the process of mathematization

4 Summary and Future Issues

Wittmann declares that the important results of research in mathematics education are sets of carefully designed and empirically studied TU that are based on fundamental theoretical principles.(1995,p.365) This declaration induced our research to investigate his thought and design-principles on TU and to clarify the point at issue on the practice of TU in the classroom We proposed the new design-principles where Dorfler's generalization model was added in The teaching material "Star Pattern" was designed, practiced, and evaluated in terms of the new principles We showed its effectiveness in this report

The accumulation of research on a large number of special groups could make the progress of group theory itself, said Wittmann He indicates the research of TU may have the same story as the group theory in his paper of "Mathematics Education as a 'Design Science' "In a similar way, the detailed empirical study of a large number of substantial TU could prove equally helpful for mathematics education."(1995,p.365) The new principles reinforced by Dorfler's generalization model could be effective not only to the description but also as the normative model of the teaching process in TU In other word, TU are the fruitful target for meta-cognitive research On the contrary, TU can gain educational significance through such researches

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THE TEACHING-RESEARCH DIALECTIC IN A MATHEMATICS COURSE IN PAKISTAN

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Abstract

This study is about teachers learning mathematics in a six week course in Pakistan. The course was research based and used a unified approach to pedagogy and content. The paper focuses on the learning of quadratic functions through a multi-representational approach linking algebraic and graphical forms. It draws on significant episodes in the learning process, using as evidence classroom observations, interviews with teachers, and teachers' and researchers' reflections. It points to the development of metacognitive awareness of the teachers in both mathematics and pedagogy. An aim of the research was to feed experience from the course into the design of materials for teachers at a distance.

The programme

Teachers in a Masters' programme (MEd) in Karachi, Pakistan, were asked to choose a subject discipline in which to enhance their 'content' knowledge. Six of them chose to study mathematics. They came with a range of experiences. The two with most mathematics were lower-secondary teachers; two taught some mathematics at primary level; the other two did not teach mathematics. In their own schooling, all had followed the standard mathematics curriculum which was text-book based. Teachers generally follow the text book closely and students are expected to follow the examples precisely as given. The teachers acknowledged that they had experienced largely rote learning, and that their own teaching had encouraged this too. However, in their MEd programme, which had overall objectives of cooperative learning and (what they call) 'friendly argumentation', they had participated in a module in mathematics education which had emphasised mathematical processes in problem-solving and investigational work with a strong metacognitive element. Because they had enjoyed and been inspired by this module they chose to study mathematics further.

This study involved a six-week course in mathematics which was tailored to fit the students' needs. The course was designed to address common elements in the mornings, and to encourage students' individually chosen study, with tutor support, in the afternoons. As well as the 6 MEd students, 5 other teachers, all more experienced mathematicians, chose to join the course on the understanding that it would be focused principally for the MEd students. They wanted the opportunity to study mathematics for a concentrated period with experienced tutors.

The course tutors came from Oxford, which has a partnership agreement with the Aga Khan University in Karachi and contributes to selected programmes in a collaborative arrangement. This would be the first time that this course had been offered. It presented an exciting opportunity for an innovative programme based on the enhancement of the teachers' conceptual understanding of mathematics in chosen areas. It was designed to weave pedagogy and mathematics, so that students could reflect on their mathematical learning and consider implications for their own teaching. Students were required to keep a log of their mathematical thinking and write a journal of their reflections generally.

The research

The tutors decided to conduct research alongside the teaching for a number of purposes:

1. To study the teachers' responses to the programme and gain insights into the value of the course - in essence a course evaluation alongside its teaching;
2. To gain insights into the students' learning and to increase our wider understanding of teaching in certain areas of mathematics;
3. To gain insights into the learning of mathematics in Pakistan which would feed into the design of materials to be used in developing the mathematical knowledge of teachers at a distance in Pakistan and beyond.

The research was conducted over the six weeks, but two of those weeks were given a concentrated attention. During this time, two tutor-researchers worked with the 11 participants in the group. Mainly, one was 'the teacher' and the other was both a participant observer and interviewer. The teacher managed the sessions and worked with the students on mathematics, either as a whole group or as small groups and individuals. The observer recorded events during whole group sessions, and interviewed participants during self-study sessions.

The methodology was, largely, to identify incidents significant in the mathematical learning of the students and to gain as much information as possible regarding these incidents. Incidents were identified in a number of ways: for example, by consensus within the group; by one or both researchers in discussion about what had occurred; or by one or more participants in interview or conversation. Data consisted of hand-written notes, audio recordings which were then transcribed, excerpts from students' journals which they shared voluntarily with researchers, students' more formal writings – either as required for assessment purposes, or written for other reasons – and researchers' reflections. Analysis, which is still taking place, involves a close scrutiny of the data with organisation and categorisation of emerging insights, and triangulation where possible between different sources of data.

The mathematical foci during the two weeks of concentrated study were *functions* and *geometry*. This paper focuses only on functions.

Theoretical perspectives

The course planners were aware that the teachers involved had a very mechanistic experience of algebra which was replicated in their teaching. The course was designed to enable teachers to address algebraic roots through a consideration of *functions*, aiming for relational understandings based on multi-representational approaches. The ultimate aim was that this would lead to teachers using more conceptual approaches to children's learning of algebra in their own classrooms.

Dreyfus and Vinner (1989), in their application of the *concept-image – concept -definition* schema on the learning of functions, suggest that concept images are not simply formed by definitions but by experiences. This explains the diversity of concept images associated with the concept of function: a correspondence between two variables; a rule of correspondence; a manipulation or operation; a formula/algebraic term/equation/graph. Additional diversity and refinement of these images is suggested by the students' images when given graphs of functions (see also Barnes, 1988; Markovits et al 1986; Vinner, 1983).

A major source of learners' difficulties with functions is their lack of flexibility in switching representations or working on the relationships between them. Dreyfus and Eisenberg (1983) demonstrated evidence of such difficulties. In sum, students tend to view algebraic representations and graphical representations as being independent. Their major difficulty seems to be to shift between different representations of functions and to establish connections between static (object) and dynamic (process) aspects of its nature (Sfard, 1991).

Remarkably similar tendencies are evident in the behaviour of prospective mathematics teachers (Even, 1993): in her study, the function is seen as always defined by a formula or a continuous, and 'reasonable' graph but scant reference is made to links between the two. And, as Even claims, mathematics teachers with fragmentary and disconnected concept images of functions are not likely to engage in flexible multi-representational teaching.

The following evidence is emerging from the project, some of which will be exemplified below:

1. the teachers' learning of functions as a gradual journey towards the ability to switch between different modes of representation: equation, formula and graph. Their transition from a mechanistic manipulation of formulae to an understanding of the algebraic and graphical meanings of these manipulations;
2. a unified approach in terms of pedagogy and content, which took an elemental learner-inclusive focus to mathematical concepts, reaped rich rewards in terms of teachers' responses as in (1) and subsequent approaches to their own learning in other areas of mathematics;
3. a metacognitive awareness, deriving to some extent from the research-related nature of the course, contributed to teachers' growth of confidence in their ability to make mathematical sense, conduct their own mathematical study, and work in more conceptual ways with their own students.

We offer a flavour of the above evidence with reference to a particular significant event. Its significance was of group consensus, being recognised at the time it occurred and later in individual interviews and journal records.

An exemplary case: significance in multiplying a quadratic equation by -1

We had worked together on finding where graphs of quadratic functions 'cut' the axes. It had been established that this occurs when either x, or the function is zero. This gave a purpose to addressing solutions of quadratic equations, and provided a context for linking algebraic and graphical representations. Several of the teachers were very fluent in applying solution methods of factorisation, completing the square and quadratic formula. They acknowledged that these were based on memorisation, and they could say little about how the methods worked or about their origin. At a particular point in the whole group discussion, the task was to solve the equation $-2x^2 + 3x + 9 = 0$ by the most efficient method possible. We continue with the words of one of the less experienced teachers, Sherwin, as written in his journal.

> "I intervened and requested a volunteer to solve the function using the quadratic formula as the negative coefficient of x^2 was leaving me somewhat sceptical as to its solution. The tutor acceded to my request and a volunteer endeavoured to solve the equation during which process, one of the steps he did was to multiply the same by -1. At the end of the solution of $-2x^2 + 3x + 9 = 0$, a member of our Math group [Sikunder] asked, "Will multiplying the equation by -1 change its function?", following which a discussion ensued as no satisfactory answer was then available. Finally the tutor intervened and using the blackboard as a teaching resource, drew parabolas of the original equation and the amended one. For me that was the precise moment when I grasped what was being said."

The teacher continues with a "reflective analysis" on the event he had described. This analysis includes the following excerpts:

> "I just took it for granted that this operation would have no effect on the outcome (i.e. behaviour of the graph) and consequently the function would remain the same.
>
> The reason for my thinking in this manner was that a mathematical process involuntarily triggered off in my mind with reference to balancing an equation.
>
> The moment when [the tutor] pictorially depicted the functions of the two equations on the same graph, I realised "Goodness, it's the exact opposite image of the original parabola."
>
> I conceptualised what in mathematical language is described as a reflection of the original function $f(x) = -2x^2 + 3x + 9$ in the x-axis."

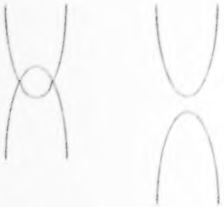
He ends with consideration of "factors that led to my understanding" These include: "[The tutor's] flexibility in that she granted my request and compromised on the time, but not on an individual learner's understanding; a group member's question, "will multiplying by -1 ... ?", thus ensuring thinking aloud and subsequent discussion between the Math group and the tutors; the tutor's questions "what do you notice about these two graphs?"; the sketches of these two functions made by [the tutor] on the board and ultimately my actually doing the same thing on my calculator."

Later in the same session, we had started to consider another function. We use the words from one of the researchers in describing what occurred:

> "A highly significant moment (for both researchers and teachers, it emerged) arose later in the session. Discussion had moved on to the nature of the roots of an equation and its relationship to the graph of the function. The equation $x^2 - 2x + 2 = 0$ was shown to have no real roots. The graph was drawn of the function $f(x) = x^2 - 2x + 2$, showing an upright parabola not crossing the x-axis, i.e. with its minimum value above the axis. The tutor asked for an example of a function with a maximum value below the axis. As quick as a flash, one of the less experienced teachers, Sherwin, suggested just altering the signs of f(x), i.e. giving $-x^2 + 2x - 2$. The tutor, and several other teachers gasped at this response. It seemed physically to portray this teachers' meaningful perception of the theory behind the -1 event.
>
> Later, I interviewed Sherwin and two other teachers about the significant instances for them of the class session, and its focus. All spoke of the -1 event and Sherwin's later response. Sikunder, who teaches mathematics at A'level, indicated that although he 'knew' in theory the relationship between roots and coefficients of an equation and their graphical representation, he had nevertheless for the first time realised the significance of the function and its inverse crossing the x-axis at the same two points. He felt that Sherwin had made a conceptual leap in translating this position to one where the graph did not cross the x-axis, seeing the generality of the inversion of the function through multiplication by -1."

As part of her reflection on this event, the researcher wrote:

> "My initial impression was that Sherwin's suggestion was triggered by his realising how the coefficients a, b and c of the quadratic $ax^2 + bx + c$ relate to its graphical representation (in this case $a>0$ implies that the parabola is 'upward' and $a<0$ implies it is 'downwards'). This impression was grounded on the evidence of the class discussion of the roles of a, b and c while solving equations [relating to three other functions discussed earlier in the session].
>
> "My impression was altered during the interviews. Sikunder highlighted his perception of Sherwin's thinking as follows: Sherwin may have known the role that the coefficient *a* plays in the graphical representation of a quadratic function. However, in this case the discussion on multiplying by -1 to obtain a quadratic with the same roots but a reflection of the original one led him to think of a way of obtaining a quadratic with no roots and a maximum from a quadratic with no roots and a minimum. In other words the picture below on the left led him to think of the picture in the right



> "Sherwin in his interview confirmed Sikunder's account. Significantly in the interviews a number of the teachers mention this incident as a poignant learning moment that had occurred during that morning's session."

Discussion

We have presented this episode in the above format to emphasise the methodology of interweaving and encouraging teachers' metacognitive involvement in mathematics and pedagogy, and the reflections of all concerned in providing evidence to support significance in the research. We shall discuss issues arising from the episode and relate them to those emerging more broadly as analysis continues.

Firstly, Sherwin's intervention and Sikunder's question indicate the interactive nature of whole group sessions. All participants were encouraged to contribute, ask questions and suggest ways forward. The "volunteer" who multiplied by -1 to solve the equation $-2x^2 + 3x + 9 = 0$ was one of the less experienced mathematicians in the group. It became accepted that feeling uncertain should not be a barrier to contribution, and that mistakes were a valuable stage in conceptual development. There are many examples in the data which show that this was the case.

Given the above approach, tutor input was just one part of the wider discussion. However, the greater knowledge and experience of the tutor brought with it a responsibility. Being aware of the nature of rote learning in most schools, and the Pakistani respect for and deference to those in positions of authority, the tutor had to exercise with caution a temptation to launch into explanations. Discussions between the researchers after sessions dwelt frequently on ways in which tutor input seemed to encourage or inhibit mathematical development in the group. In this case, for Sherwin and others, the input seemed to be felicitous in promoting useful images. However, there were other cases where it was judged less favourably.

Mathematically, there was at least one important insight from this episode. Sherwin expected that multiplication by -1 would not change the function, relating this to what he knew about "balancing an equation". The language used was occasionally problematic. For example, in the first line quoted from Sherwin, he says "... volunteer to solve the function using the quadratic formula ... ". Did he mean here "the function", or was this a language slip? If he meant it, did this imply that he thought it possible to 'solve a function'? If so, what did this mean, for him? Was he confused by the difference between an equation and a function? Is this a source of wider confusion for students? Later he seems to see the difference, when he comments on the "reflection" of the original function in the *x*-axis, and moreover when he makes the conceptual leap to a different function. Has the conceptual development here allowed him to become clearer about the relationship between functions and equations, as well as between the algebraic representation and the graphical one?

For more experienced mathematicians this episode also had mathematical significance. Sikunder, an A'level teacher, acknowledged his own developing concept. Others indicated similar experiences. Participants of widely varying experience worked together for mutual benefit, the more experienced recognising new images and perspectives emerging from their work. Again, we have considerable evidence from observations, interviews and participants' writing.

Sherwin's "reflective analysis" and "factors that led to my understanding" exemplify the metacognitive element of teachers' thinking. This metacognition was a result not only of this course per se, but also of the ethos of the MEd course as a whole, where students were overtly required to reflect and to write their reflections. Whatever the basis, however, the quality of this reflection is evident in Sherwin's analysis. Such quality extended to all participants. We were overwhelmingly impressed by the quality of insights both mathematical and pedagogical which emerged during the research.

The research was an important element in the developments which took place. After early reservations were overcome, participants welcomed individual interviews as opportunities to talk to the tutors about their developing thinking. These interviews, as other researchers have observed (e.g. Steffe, 1983), were a formative element in the teaching of the course. Teachers gained confidence from their participation in a research project. They offered, willingly, their written reflections for the tutors to read and provide comments. They also exchanged reflections and wrote comments for each other, a normal routine in the MEd course.

One final remark at this stage: an objective of the research was to gain insights into the learning of mathematics in Pakistan which would feed into the design of materials to be used in developing the mathematical knowledge of teachers at a distance. Such insights are many and varied. However, an issue which arises concerns a tension between the open, participatory nature of this course, and the need to provide some measure of closure in materials for distance learning. For example, given the importance of conceptual understanding arising from the -1 episode, materials could valuably include some work around this concept. But, how should it be introduced? What should students be asked to do? How might comments be provided? Can such a 'AHA' moment be anticipated/replicated? When AHA moments occur, at a distance, where is the experienced tutor to offer just the level of input which can be fruitful? How is a metacognitive element developed at a distance? We are grappling with these issues currently as we start to produce materials alongside the other participants.

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CHARACTERISING MATHEMATICS TEACHING USING THE TEACHING TRIAD

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This paper reports ongoing research which attempts to make sense of the complexity of mathematics teaching at secondary school level. The research was conducted in partnership between two teachers and two researchers over one school term in two U.K. schools. A theoretical construct, the Teaching Triad, was used both as an analytical device (by the researchers) and as a reflective agent for teaching development (by the teachers). The focus here is on the methodology of the project, exemplified through substantive issues which emerged.

Background

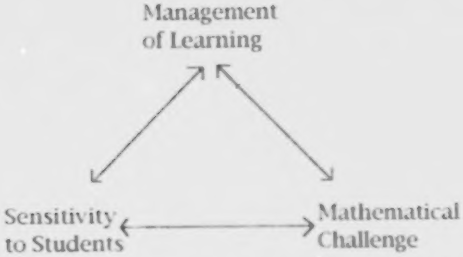
Our starting points in exploring classroom mathematics teaching are:

1. that the complexity of teaching-learning situations is such that any attempt to provide holistic descriptions is rapidly shown to be hollow (Bauersfeld, 1988);
2. that the essence of teaching/learning processes lies in classroom interactions, the analysis of which provides the key to understanding the complexity acknowledged in (1);
3. that teachers work from a position of sincerity and professionalism in seeking for students' mathematical development.

The research mainly fits Wagner's (1997) second category of researcher-practitioner cooperation, "clinical partnership", in which the researchers are the main agents of enquiry and the practitioners those whose work is under scrutiny. However, in this research teachers are also consultants in the analytical process, and their perceptions are central to research analyses. The basis of our methodology is the close scrutiny of classroom lessons and teachers' thinking motivating those lessons. This involves a *micro*analysis of classroom interactions, and a related *macro*analysis drawing on the thinking and reflections of participants including students, teachers and researchers. The research takes place as part of an evolving tradition which recognises the classroom as a social setting and analyses teacher-student interactions as central to the constitution of this setting (e.g. Cobb, Wood & Yackel, 1992; Steinbring, 1993; Jaworski, 1991; Simon, 1995; Voigt 1996).

The Teaching Triad

The *Teaching Triad* emerged from an ethnographic study of *investigative mathematics teaching* by one of the authors (Jaworski, 1991). It describes or characterises the *teaching* in a classroom, attempting to provide a framework to capture the essential elements of the complexity involved. It encompasses three domains: the management of learning (ML); sensitivity to students (SS) and mathematical challenge (MC).



Briefly, ML describes the teacher's role in the constitution of the classroom learning environment by the teacher and students. Thus it includes the classroom groupings; planning of tasks and activity; setting of norms and so on. SS describes the teacher's knowledge of students and attention to their needs; the ways in which the teacher interacts with individuals and guides group interactions. MC describes the challenges offered to students to engender mathematical thinking and activity. This includes tasks set, questions posed and emphasis on metacognitive processing. These domains are closely interlinked and interdependent, although their dependence can be seen in a variety of ways, which is one object of study in the current research.

The current research

During an earlier research project, the two teachers had expressed, independently, an interest in the teaching triad as a useful device in helping them to reflect on and improve their mathematics teaching (See, for example, Jaworski & Edwards, 1997). Although it had emerged from the context of *investigative* teaching, the teaching triad was now to be tested against teaching which might or might not be described as investigative. Thus it was hoped both to use the triad to gain insights into the observed teaching, and also to test out its applicability to teaching more broadly.

The two teachers designed and taught lessons to a number of classes of students and reflected on practices and outcomes. The chosen classes encompassed a range of ages and abilities in Years 7 to 12 (ages 11 to 17). Lessons were observed by one or both researchers who recorded interactions in classroom notes and audio or video recordings. Teachers and researchers talked at length, about the lessons and issues arising from them, both in school and as a foursome out of school. Objectives of the participants were complementary, but not the same. For the teachers, a major objective was to gain insights into their own practices in order to develop or improve teaching. Researchers sought the nature of the practices observed, issues arising and theoretical conceptualisations of the teaching.

It is important to recognise, in this research, that lessons do not follow a set format and there are no standard forms of interaction. While there are some patterns in approaches used and interactions observed, there are nevertheless many different styles of lessons depending on the particular objectives on which a lesson is based. Thus, any episodes quoted as examples should not be seen as generic of the teaching as a whole. Rather they will be used to provide insights into practices; issues and research analyses. However, we take seriously the words of Cooney, who writes

> *But if we are to move beyond collecting interesting stories, theoretical perspectives need to be developed that allow us to see how those stories begin to tell a larger story. That is we should be interested in how local theories about teachers can contribute to a more general theory about teacher education. (Cooney, 1994, p. 627)*

Ultimately we are interested in how our approach, using the teaching triad, can provide a means for contributing to a more general theory of describing and interpreting teaching practice, and ultimately to indications for teacher education. In this spirit we offer, below, analysis of a lesson which will exemplify our approach, and indicate how this approach leads to theory generation. First, however, we address, briefly, areas of research to which this study is related.

A number of researchers have investigated mathematics teaching by considering its effect on students' cognitive development. Moreover, they have examined the development of the teaching when the emphasis is on students' learning (See for example, Wood, Cobb, Yackel & Dillon, 1993; Steffe & Wiegel, 1992). Steinbring (1997) investigated teaching by considering the development of mathematical meaning through an interplay between social constraints of the communicative process and the epistemological structure of the mathematical knowledge. The studies of Cobb, Yackel & Wood (1989) and a number of studies reported by McLeod (1994) (the work of Williams and his colleagues e.g.. Williams & Baxter, 1993) analyse the relationships among discourse, learning, and affect in mathematics classrooms. Other studies consider the role of the wider social context on introducing innovations in mathematics teaching (see for example Dillon, 1993). These examples highlight different focuses in related research on mathematics teaching. Hoyles (1988) emphasises the need to develop tools of analysis of the teaching/learning situation so that to cope with the complexity without "focusing solely on cognitive aspects, on attitude of the teacher, on classroom practices, and so forth". By using the theoretical construct of the teaching triad, we aim to address this complexity.

Exemplifying the analytical process: Lesson: *Packing Cubes*

Episode 1 - Lesson's opening

Teacher A's class was divided into groups of 4, 5 or 6 students. She gave each group sheets of squared paper, some card, and a sheet describing a problem. Multilink cubes were also available for use. The problem was to design a box to contain 48 cubes, each of side 2 cm, using a minimum amount of card. Each group had to work on this task and provide a group solution. The lesson opened with an interactive whole class session in which the task was clarified. She invited someone to come to the board and draw a 2cm cube. Michael accepted the invitation and drew a 2cm square. Some students said that it was a square (i.e. not a cube). The teacher acknowledged it as "a cube facing the front". Another student came forward and drew a cube. The teacher asked him "Can you write your dimensions as Michael has done?"

The teacher then clarified the problem: "You need to make a box to fit 48 of those cubes. The squared paper is to try out some designs. It is a group project. Everyone needs to be busy." She paused, waiting for silence in the room. "The company will give a prize to the one who has used the least card. We want to find the best shape and size of box".

In our analysis, it was not difficult to fit the triad to this opening. In the category of ML, we recognised the teacher's organisation of the class into groups (a regular feature of her lessons); her planning and organisation of the task; her management of the whole class situation in which she introduced the task; and her responses to the students who participated. All of these are part of the teacher's organisation of the learning environment, her developing of social skills within the classroom setting, and her creation of opportunity to engage in mathematics.

In the category of SS, we placed her mode of introduction of the task, and her particular responses to students. She wanted all students to be able to make a start on the task, hence she offered a practical context, resources, and clarification of the problem. She engaged students in her introduction and was careful to value, and encourage others to value, their contributions.

As well as valuing Michael's contribution, her words "a cube facing the front" acknowledged another perspective of what students saw as a square. Thus she opened up mathematical challenge (MC), initially manifested in the problem itself, and enhanced through her contextual emphasis on optimisation in the box.

Episode 2 - Interacting with a group of students

We focus on a subsequent interaction between the teacher and two boys, Tom and Stewart. The boys had two different organisations of the 48 cubes: Tom with 48 cubes in a line; Stewart with a 2x4x6 ($4 \times 8 \times 12 cm^3$) cuboid. They had each drawn nets of their solids to enable them to calculate surface area (respectively $776 cm^2$ and $352 cm^2$), and had worked out the volume ($384 cm^3$). The teacher looked at these two cases with the boys, and it was acknowledged that, although their volume was the same, Tom's surface area was greater than Stewart's. Tom's contribution to the discussion, seemed to indicate his understanding of the situation. At this point, Stewart's understanding was less clear.

The dialogue below followed immediately. One remark followed quickly on another and often the participants were all talking at the same time. (T: Teacher; To: Tom; S: Stewart):

1 T. [To Tom] Brilliant, Good. That's right. OK. But, by putting them like this, – which you knew was going to be bigger because (contributing voices drown her words) so, your net's going to be 776 cm^2 and Stewart's net's going to be only 352cm^2. Now I want you to think why Stewart's is less.
2 S. Cause mine's higher and wider and
3 To It's easier to fit in the trolley. [He refers to a supermarket trolley]
3 T. It is easier to fit in the trolley, yes.
4 S. Because mine's got more height and width than Tom's.
5 T. Right, so it will be (many voices) consequently its been made
6 S. Shorter
7 T. Shorter, (many voices)
8 S. More compact
9 T. Right. *Compact.* Good. Right now, is Stewart's model the most compact model you can come up with, or is there anything better? Well. I don't know. Let's look at James and John's to see if they've done better.
10 [Some interruption here from other students to whom the teacher responds]
11 T. Stewart, you have done really well so far, OK?
12 S. Yes.
13 T. But you need to make sure you are listening in to the other's designs as well.

A triadic microanalysis shows considerable complexity here.

1. Praise for Tom: teacher values his thinking and contribution (SS). The teacher offers a refined challenge (MC) based on students' immediate thinking (SS).

2-8 Teacher's questions lead to Stewart's articulation of his perspective. Despite seeming ambiguity of words chosen, the teacher seems satisfied that Stewart understands the position. His use of the word 'compact' seems particularly important in capturing the concept which the teacher seeks. A complexity here of ML and SS, as the teacher manages the situation with questions which impose limits on the dialogue, and allow Stewart to express what he sees.

9 Teacher's emphasis of key word, 'compact', values Stewart's articulation (SS). Teacher uses Stewart's language to pose a further challenge: (MC). She emphasises the value of group sharing and negotiation: (ML).

10 Attention to other students and classroom norms (ML)

11-12 Praise for Stewart (SS) which Stewart acknowledges.

13 Reminding Stewart of the importance of collaboration (ML).

Within the category of *management of learning* we see

1. the created environment: including classroom groupings, the set task, and expectations of a) collaborative working, b) expression of mathematical thinking, c) students' autonomy in their activity;

2. her interactions with individuals and groups in which her questions focus students' attention conceptually and encourage their own articulations, balancing teacher input with student autonomy;
3. her reminders of classroom norms and requirements of the set task.

Sensitivity to students is manifested in her attention to individuals or groups: finding out what they have done, gaining access to their thinking, praising their achievements, judging responses to them relative to her conceptually-based objectives.

Mathematical challenge appears evident in the task itself, in expectations that individuals will direct their own activity (e.g. in choosing their own organisations of the cubes to explore), and in focused questions to enable concepts to be tackled.

The interdependence of these categories can be seen in terms of sensitivity needing to fit with challenge and vice versa – challenges without appropriate sensitivity might fall on stony ground – and in the construction of the learning environment which provides opportunities and supports participation.

At the macro level, the teacher has a clear agenda about what she wants students to achieve. Through a freedom to select their own organisations of the 48 cubes, and a comparison of different organisations within a group, she wants them to recognise same volume, but different areas, and subsequently to seek the minimum area. Providing freedom requires that her further challenges are designed to fit students' thinking. She encourages articulation of their current conceptualisation which allows them to inform her of their thinking and potentially to clarify the thinking.

In planning this particular lesson, the teacher drew a diagram with the three elements of the TT and tried to fit her planning to this diagram. First, she started to consider the kind of mathematical challenge that she could offer to the students. Then she moved to the management of learning where she considered the resources she needed to offer, the classroom organisation, the encouragement of communication in the group. The sensitivity to students was seen as a way of providing students the chance to explore, discuss, argue but also to encourage building of confidence on their own ideas. Sensitivity was also seen by the teacher as the base for further mathematical challenges meaningful to students.

The episode, both in the opening of the lesson and in the teacher's interaction with the two boys, shows close harmony between sensitivity and challenge in the cognitive domain within a management of learning which values students' activity and thinking, and emphasises collaborative work in supporting concept construction. A conjecture is that teaching is most effective when these three categories seem at their most harmonious.

Sensitivity and Challenge: The Cognitive and the Affective

While mathematical challenge can be seen to operate largely in the cognitive domain, sensitivity seems to work in both the cognitive and the affective domains for the mathematical learner. A clear characteristic of Teacher A's teaching is her concern for affective elements in her classroom. This concern goes alongside the cognitive dimension of her teaching as may be seen in the lesson opening. She was concerned as much for Michael's self-esteem, as for his cognitive development of the concept of 'cube'. She is overtly concerned about students' 'self-esteem' in her lessons, which was the subject of her own previous research into her teaching (Jaworski & Edwards, 1997). This might be seen as a factor principally of an *affective* nature. It concerns students' personal belief in, and valuing of, their ability to do mathematics and think mathematically. It concerns students' wellbeing within the classroom setting, and this does not necessarily include a cognitive dimension. Sometimes, for Teacher A, it was important to question the direction of affective considerations and tie them to cognition in order to judge classroom episodes as effective in learning terms. The balance between affect and cognition was one of the improvements she wanted to make in her teaching in this particular class. She chose to study her teaching in this class because she felt that she probably didn't offer to the students the cognitive challenge they needed although they were happy and had developed good relationship with her. She wanted to "push them further".

An analysis which attends to cognitive and affective elements and their relation to sensitivity and challenge in the teaching triad has potential to be informative to teaching development. This relation started to emerge from observing and analysing teacher A's teaching and from the discussions with her. At a subsequent level, it provided the teacher the opportunity to analyse further how she conceived her sensitivity to the students both in planning her lesson and in reflecting on it. Further analysis of her lessons is needed to see whether this awareness actually led to a development of her teaching.

We have been unable in the space of this paper to address the teaching of Teacher B. The two taken together offer important similarities and counterpoints which enable us to subject the research to a more critical scrutiny and provide a wider perspective on the teaching triad. For example, episodes from both teachers in which the degree of harmony between sensitivity and challenge is considerably less that in the example quoted in this paper allow relationships within the triad to emerge more clearly. Ways in which the teaching of the two teachers is characterised differently through the triad provide, again, valuable evidence for its analytical and descriptive power.

In conclusion

Microanalysis reveals the finer points of interactions which highlight the issues for a teacher in creating challenges and responding to students' needs. Challenge and sensitivity need to be in harmony, but what are the elements of this harmony? Analysis is starting to show certain patterns in interactions which will lead to general propositions which can be tested further by these teachers and others. Already indications of ways in which the triad interfaces with cognitive and affective dimensions of teaching are proving fruitful in gaining insights into effective interactions. Teachers' reflections on episodes offer counterpoints to researchers' analyses which remain to be developed further. Ultimately, the theory emerging from this research will be in the form of indicators which can be tested in other classrooms.

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The Mediation of Learning within a Dynamic Geometry Environment

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Computer-based learning environments promise much in terms of enhancing mathematics learning. Yet much remains unclear about the relationship between the computer environment, the activities it might support, and the knowledge that might emerge from such activities. The analysis presented in this paper is offered as a contribution to understanding the relationship between the specific tool being used, in this case the dynamic geometry environment Cabri-Géomètre, *and the kind of thinking that may develop as a result of interactions with the tool. Through this analysis I suggest a number of effects of the mediational role of this particular computer environment.*

Introduction

Computer-based learning environments continue to be a seductive notion in mathematics education. As Balacheff and Kaput (1996 p469) explain, it is possible for such environments to have an intrinsic cognitive character which is unique when compared to other learning materials. This means that such environments may be able to offer "a channel of access to the world of formal [mathematical] systems" (Noss 1997 p30). The promise is that through using particular software in carefully designed ways, it is possible simultaneously to use and come to understand important aspects of mathematics, something that in other circumstances can be particularly elusive.

One type of promising environment identified by Balacheff and Kaput (1996 p492) features what is commonly referred to as the "direct manipulation" of mathematical objects and relations. Prime examples of this type of software development are dynamic geometry environments (DGEs), such as *Cabri-Géomètre*. Yet, as diSessa *et al* (1995 p2) point out, significant issues with important practical ramifications remain under-researched. A vital question, and the theme of this paper, concerns the relationship between the specific tool being used and the kind of thinking that may develop as a result of interactions with the tool.

This paper explores some early findings from a longitudinal study aimed at investigating how using the dynamic geometry package *Cabri-Géomètre* mediates the learning of certain geometrical concepts, specifically the geometrical properties of the 'family' of quadrilaterals. In what follows I suggest some aspects of the mediational role of the DGE *Cabri-Géomètre*. I begin by outlining the theoretical basis of this view of tool mediation.

Theoretical Framework

From a sociocultural viewpoint, the development of mathematical reasoning is viewed as culturally mediated, through language and through the use of artifacts, both of which are referred to as tools. For example, Rogoff (1990 p13) argues that "children's cognitive development must be understood not only as taking place with social support in interaction with others, but also as involving the development of skill with socio-historically developed tools that mediate intellectual activity". Such a perspective builds on the work of Vygotsky who stressed that "the central fact about our psychology is the fact of mediation" (Vygotsky 1982 p166). Wertsch too extends Vygotskian ideas by incorporating elements from the work of Bakhtin (Wertsch 1991, Wertsch *et al* 1995) demonstrating how language is a mediating influence which "lies on the borderline between oneself and the other .... [so that] expropriating it, forcing it to submit to one's own intentions and actions, is a difficult and complicated process" (Bakhtin 1981 pp 293-294). As Cobb (1997 p170) confirms "tool use is central to the process by which students mathematize their activity", and further that "anticipating how students might act with particular tools and what they might learn as they do so is central to our attempts to support their mathematical development"

Such theoretical work suggest some elements of tool mediation which can be summarised as follows:

1. Tools are instruments of access to the knowledge, activities and practices of a community.
2. The types of tools existent within a practice are interrelated in intricate ways with the understandings that participants in the practice can construct.
3. Tools do not serve simply to facilitate mental processes that would otherwise exist, rather they fundamentally shape and transform them.
4. Tools mediate the user's action - they exist between the user and the world and transform the user's activity upon the world.
5. Action can not be reduced or mechanistically determined by such tools, rather such action always involves an inherent tension between the mediational means and the individual or individuals using them in unique, concrete instances.

Examples of mathematics education research which make use of the notion of tool mediation include Cobb's study of the 100 board (Cobb 1993), Säljö's work on the rule of 3 for calculating ratios (Säljö 1991), and Meira's examination of using gears to instantiate ratios (Meira 1995).

Applying such notions to learning geometry within a DGE suggests that learning geometrical ideas using a DGE may not involve a fully 'direct' action on the geometrical theorems as inferred by the notion of 'direct manipulation', but an indirect action mediated by aspects of the computer environment. This is because the DGE has itself been shaped both by prior human practice and by aspects of computer architecture. This means that the learning taking place using the tool, while benefiting from the mental work that produced the particular form of software, is shaped by the tool in particular ways.

Some aspects of the mediational impact of *Cabri-geomètre*

Applying this theoretical perspective to the implementation of *Cabri I* for the PC, together with a reading of previous research (for example: Goldenberg and Cuoco 1998, Hölzl 1996, Jones 1996, 1997), leads to the following being identified as possible mediating influences on learners using this particular tool:

1. The layout of the interface with the separation of the *creation* and *construction* menus.
2. The default cursor operation, which is drag rather than, say, the creation of a point.
3. The existence of a number of different forms of point in *Cabri I*: basic point, point on object, (point of) intersection - not to mention midpoint, symmetrical point, and locus of points , plus centre of a circle; also rad pt (radius point) and circle point.
4. The existence of a number of several forms of line: basic line, line segment, line by two pts (points) - not to mention parallel line, perpendicular line, plus perpendicular bisector, and (angle) bisector.
5. The existence of two different forms of circle: basic circle, circle by centre & rad pt.
6. The implementation of the drag-mode within the software which entails decisions being made about the behaviour of the geometrical objects when they are dragged. For example basic circle and circle by centre & rad pt behave differently under the drag-mode.
7. The fact that some points can be dragged while others cannot. For example, constructed points such as a (point of) intersection or a midpoint cannot be dragged, while others, such as a point on object can be dragged but only in a particular way.
8. The behaviour of a point placed arbitrarily on a line segment when an end-point of the segment is dragged.
9. The sequential organisation of actions in producing a geometrical figure. This implies the introduction of explicit order where, for most of the users, order is not normally expected or does not even matter. For example, *Cabri-géomètre* induces an orientation on the objects: a segment AB can seem orientated because A is created before B. This influences which points can be dragged and effectively produces a hierarchy of dependencies in a complex figure.

None of the above is necessarily a criticism of *Cabri*. In the implementation of such software, decisions have to be made. The point is that *the decisions that are made mediate the learning*. The remainder of this paper documents some examples of this shaping of learning within this particular DGE in an attempt to reveal possible tensions between the tool and the actions of the learners.

Empirical study

The empirical work on which the observations below are based is a longitudinal study examining how using the dynamic geometry package *Cabri-géomètre* mediates the learning of geometrical concepts. The focus for the study is how "instructional artifacts and representational systems are actually used and transformed by students *in activity*"

(Meira 1995 p103, emphasis in original) rather than simply asking whether the students learn particular aspects of geometry "better" by using a tool such as *Cabri*.

The data is in the form of case studies of five pairs of 12 years old pupils working through a sequence of specially designed tasks requiring the construction of various quadrilaterals using *Cabri-géomètre* in their regular classroom over a nine month period. The version of *Cabri* in use was *Cabri I* for the PC. Sessions were video and audio recorded and then transcribed. In all, over 40 lesson transcripts are being analysed in two phases. The first phases identified examples of tool mediation, a number of which are illustrated below. The second phase, currently in progress, is intended to track the genesis of such tool mediation of learning.

Examples of tool mediation

Below are four examples of extracts from classroom transcripts which reveal aspects of the tool mediation of learning within the dynamic geometry environment.

Example 1

Pair Ru and Ha are checking, part way through a construction, that the figure is invariant when any basic point is dragged

Ru Just see if they all stay together first
Ha OK
Ru Pick up by one of the edge points [H drags a point]
Ha & Ru (together) Yeah, it stays together!

Note the pupils' use of the phrase "all stay together" to refer to invariance and the term "edge point" rather than either radius point (or rad pt as the drop-down menu calls it) or circle point (as the help file calls that form of point).

Example 2

Pair Ho and Cl are in the process of constructing a rhombus. As they go about constructing a number of points of intersection, one of the students comments.

Ho A bit like glue really. It's just glued them together

This spontaneous use of the term "glue" has been observed by other researchers (Ainley and Pratt 1995) and is all the more striking given the fact that earlier on in the lesson the pupils had confidently referred to points of intersection as just that.

Example 3

Pair Ru and Ha are constructing a square using a diagram presented on paper as a starting point. After a short discussion the pair begin by constructing two interlocking circles.

Ha If I erm
I reckon we should do that circle first
Ru Do the line first
Ha No, the circle. Then we can put a line from that centre point of the circle
Ru Yeah, all right then

Ha You can see one circle there, another there and another small one in the middle

The inference from this extract is that previous successful construction with the software package influences the way learners construct new figures.

Example 4

Pupils Ru and Ha have constructed a square and are in the process of trying to formulate an argument as to why the figure is a square. I intervene by asking them what they can say about the diagonals of the shape (in the transcript I refers to me)

Ru They are all diagonals
I No, in geometry, diagonals are the lines that go from a vertex, from a corner, to another vertex
Ru Yeah, but so's that, from there to there
I That's a side
Ru Yeah, but if we were to pick it up like that like that. Then they're diagonals

Pupil Ru is confounding diagonal with oblique, not an uncommon incident in lower secondary school mathematics. Here the software cannot provide any assistance to the student, indeed the drag facility allows any straight line to be moved to appear to the learner to have an oblique orientation. Furthermore, in terms of the specialised language of mathematics, such software can not hope to provide the range of terms required nor could it be expected to do so. Such exchanges call for sensitive judgement by the teacher.

Some observations on the examples

The examples given above are representative of occurrences within a number of the case studies. A number of comments can be made on these extracts which illustrate how learning within the computer environment is shaped by the nature of the mediating tool.

First, it appears that learners find the need to invent terms. In example 1 above, the pupil pair employ the phrase "all stay together" to refer to invariance and coin the term "edge point" to refer to a point on the circumference of a circle. To some extent this parallels the need of the software designers to provide descriptors for the various different forms of point they are forced to use. Yet research on pupil learning with Logo suggests that learners use a hybrid of Logo and natural language when talking through problem solving strategies (for example, Hoyles 1996). This, I would argue, is one effect of tool mediation by the software environment.

A second instance of the mediation of learning is when children appear to understand a particular aspect of the computer environment, in example 2 above it is the notion of points of intersection, but in fact they have entirely their own perspective. In this example, one student thinks of points of intersection as 'glue' which will bind together geometrical objects such as lines and circles. This, I would suggest, is an example of Wertsch's (1991) 'ventriloquating', a term developed from the ideas of Bakhtin, where children employ a term such as intersection but, in the process, inhabit them with their own ideas.

A third illustration of the mediation of learning is how earlier experiences of successfully constructing figures can tend to structure later constructions. In example 3 above, the pair had successfully used intersecting circles to construct figures that are invariant under drag and would keep returning to this approach despite there being a number of different, though equally valid, alternatives.

Following from this last point, a further mediation effect can be that the DGE might encourage a procedural effect with children focusing on the sequence of construction rather than on analysing the geometrical structure of the problem. Thus pair Ru and Ha, rather than focusing on geometry might be focusing rather more on the procedure of construction. This may also be a consequence of the sequential organisation of actions implicit in a construction in *Cabri-Géomètre*.

A fifth illustration of the mediation of learning within the DGE is that even if the drag mode allows a focus on invariance, pupils may not necessarily appreciate the significance of this. Thus hoping points of intersection will 'glue' a figure together, or that constructing a figure in a particular order will ensure it is invariant under drag, does not necessarily imply a particularly sophisticated notion of invariance.

From the examples given above, a sixth illustration of the mediation of learning is provided by an analysis of the interactions with the teacher (in this case the researcher). The challenge for the teacher/researcher is to provide input that serves the learners' communicative needs. As Jones (1997 p127) remarks "the explanation of why the shape is a square is not simply and freely available within the computer environment". It needs to be sought out and, as such, it is *mediated* by aspects of the computer environment and by the approach adopted by the teacher.

Concluding remarks

In this paper I have suggested some outcomes of the mediational role of the DGE *Cabri-Géomètre*. While such outcomes refer to only one form of computer-based mathematics learning environment, these outcomes are similar to those emerging from research into pupils' learning with Logo (See Hoyles 1996 p103-107):

1. Children working with computers become centrated on the screen product at the expense of reflection upon its construction
2. Students do not mobilise geometric understandings in the computer context
3. Students modify the figure "to make it look right" rather than debug the construction process
4. Students do not appreciate how the computer tools they use constrain their behaviour
5. After making inductive generalisations, students frequently fail to apply them to a new situation
6. Students have difficulty distinguishing their own conceptual problems from problems arising from the way the software happens to work
7. Manipulation of drawings on the screen does not necessarily mean that the conceptual properties of the geometrical figure are appreciated

As Hoyles remarks, such indications are intended to capture some of the general in the specific and thereby generate issues for further research. The finding from this study of the dynamic geometry package *Cabri-Géomètre* may well prove useful both to teachers using, or thinking about using, this form of software and to designers of such learning environments, as well as contribute to the further development of theoretical explanations of mathematics learning.

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USING INFORMATION SYSTEMS FOR MATHEMATICAL PROBLEM SOLVING: A NEW PHILOSOPHICAL PERSPECTIVE

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ABSTRACT

*This paper represents an ongoing exploratory formulation of an innovative theoretical framework designed to use information systems (**IS**) for mathematical problem solving. Based on theories set forth by Aristotle and Habermas, and buttressed by previous teaching experience, it is hypothesized that, in addition to the existing customary activities of data collecting and information building, the introduction of the concepts of "practical wisdom" and "emancipatory interests" in the undergraduate classroom, will greatly enhance the use of **IS** in mathematical problem solving, and add to the existing knowledge-base.*

INTRODUCTION

It is well recognized that our society is undergoing a significant transformation from an industrial to an information society, and that this transition is strongly connected with the widespread development and use of computer technology combined with the intellectual advances in such fields of mathematical inquiry as systems science, information systems, decision analysis, and artificial intelligence. This paper represents an exploratory formulation of an innovative framework designed to organize the use of **information systems (IS)** in mathematical problem solving. For the sake of clarity of discourse, "information" is defined as "an abstract concept which, when received by an individual, gives new form to that individual's perception" (Merali and Frearson, 1995). Systems dealing with information are **IS**.

Classroom experience indicates that most students have great difficulty distinguishing among data, information, knowledge, intelligence, and wisdom. Part of this difficulty stems from the fact that these five concepts are not only interdependent but also involve higher and higher levels of cognitive organizations, with each succeeding level incorporating the preceding level in a complex hierarchy. Almost without exception, students fail to use **IS** correctly because they are literally drowning in an overabundance of data and information, yet with little understanding and "practical wisdom" (Mitroff & Linstone, 1993). Machado (1980) says that every person is born with a "live computer of limitless possibilities" that can tackle the most complex problem, but unfortunately very few have the metaphorical "instructions manual" to utilize this gift. He adds that the most important job of educators today is to draw up that "manual". There is no question that in a world of rapid change and increasing diversity, the formulation of an "instructions manual" is a critical part of mathematics teaching and learning.

This ongoing research study aims to give a theoretical and philosophical account of what has been achieved and where it is heading. This is done by: introducing the concepts and components of **IS**; connecting the Habermasian and Aristotalean theories of communication and "practical wisdom" respectively, to **IS**; describing the classroom teaching experiments; and finally, detailing the ongoing research. In a broad sense, this research attempts to provide the theoretical framework of a metaphorical "instructions manual"

MEANS-ENDS CONFIGURATION

A large majority of students believe that scientific knowledge alone can be used for "solving" any real-world problem. This is partially true if the means (strategies) and ends (goals) connected with the problem are known with certainty, in which case mere mathematical computation is all that is needed to solve the problem, as shown by cell A in Figure 1. However, in almost all real-world situations, if our ends and/or means (strategies) are uncertain, mathematical computation by itself will not work. In such cases, in addition to computation, some degree of judgement, compromise, or intuition/ "inspiration" will be needed to deal with these complex, open-ended, value-laden problems, indicated by cells B, C, and D, respectively (Thomson & Tuden, 1959; Khisty, 1993).

Ends Means	**Goals & Objectives**	
	Certain	Uncertain
Certain	(A) **Computation**	(C) **Compromise**
Uncertain	(B) **Judgement**	(D) **"Inspiration" or Chaos**

Figure 1. Means-Ends Configuration

CONCEPTS OF RATIONALITY

If by rationality we mean a self-conscious process of using reasoned arguments to make and defend claims, there are indeed many choices available. The principal types of rationalities used with **IS** are connected with the means-ends framework. For example, instrumental (or technical) rationality has been used extensively, and is geared to controlling nature in an efficient and scientific way. But, as a result of reductionism, technical rationality has unfortunately been applied to human activity systems, with disastrous results. On the other hand, communicative rationality, guides communicative action, meeting the validity claims of comprehensibility, truth, rightness, and sincerity, so necessary for mutual understanding and agreement. Communicative rationality is beginning to be used in **IS**, when dealing with complex and messy open-ended problems, where the means and/or ends are uncertain. Indeed a combination of both instrumental rationality and communicative action, directed by emancipatory interests is needed for the decision making to be an enlightened social process. Emancipatory interests are important in that they reflect a concern with managing coercion and the analysis of power in organizational settings, and more will be said about this in due course (Carr & Kemmis, 1986).

THE PATH FROM DATA GATHERING TO INTELLIGENCE AND BEYOND

Now that the basic ideas of the means-ends configuration and the concepts of rationality have been set forth, the components of **IS** are mapped out, as shown in Figure 2. Naturally, any explanation as subtle as the ones described should not really start from the words themselves. Rather, on Popper's (1972) advice, "one should never quarrel about words, and never get involved with questions of terminology....What we are interested in are our real problems.."

*Data	Facts	Facts of the real-world
*Focussed Data	Selected Facts	Needed for Cognitive settings
*Information	Meaningful Facts	Needed for Problem-Solving
*Knowledge	Understanding based on information and experience	Needed for Decision-making at the Technical Rationality level
*Intelligence	Ability to deal with novel & new situations	Needed for Decision-making at the Technical & Communicative Rationality level, for discourse

Figure 2. Evolving Components of Information Systems

While most of the terms mentioned in Figure 2 are self explanatory, the meaning of intelligence needs further explanation. Intelligence is a human activity of adjusting to situations by using combinations of such functions as perception, memory, conceptualizing, abstracting, planning, extrapolating, predicting, controlling, contrasted with instinct, habit, rote, custom, and tradition. In addition, intelligence is the process of coping with problems by means of abstract thinking, containing elements such as symbolization and communication, critical analysis, applied to practical and theoretical situations (Angles, 1981). But the use of **IS** entails far more than just knowledge and intelligence, and this is taken up in the next section.

Aristotle's Theory of Practical Wisdom

It has been reported time and again that the Central Intelligence Agency (CIA) of the United States has performed poorly because of it's bent for high technology and computer models at the expense of old-fashioned, first-hand observation and common sense. One famous case reported in the 1980s states that the CIA used highly sophisticated computer models using Soviet statistics, which were known to be flawed and fabricated. The results were naturally a disaster. This example epitomizes the need for understanding and practical wisdom, and reinforces Bierce's (1967) definition of education: "That which discloses to the wise and disguises from the foolish their lack of understanding".

More than two thousand years ago Aristotle discussed his theory of practical wisdom by which the social construct system may potentially be reconstructed. In the sixth book of the Nichomachian Ethic, Aristotle distinguishes among "episteme", "techne", and "phronesis", as the three modes of the "intellectual virtues" which constitute "practical wisdom". Gadamer (1976) interprets Aristotle's ideas by saying that episteme is the intellectual virtue through which we construct our scientific knowledge, through technical or instrumental rationality. It justifies mechanistic problem-solving requiring true, efficient and unquestionable data and evidence (Flyvbjerg, 1992). However, episteme presents an incomplete picture of human and social cognition because it obscures the recognition of two other complementary cognitive virtues through which people construct their individual and social insights: their experiential knowledge (techne) and their conceptual reflective thinking (phronesis). Techne in this context should be understood in its broadest sense as "know-how". Phronesis, on the other hand, represents the cognitive virtue of reflective thinking and prudence. It is unfortunate that in the past two or three centuries phronesis, which embodies value rationality, has been totally neglected and replaced by instrumental rationality (Weber, 1958).

The practical wisdom of learning activates society's potential for coping with, and gaining from, its own diversity, as well as coping with fundamental changes in its environment by enriching and reconstructing its system of social constructs. Collective practical wisdom is the synergetic wisdom of society through recursive exchange. Gadamer contends that modern technological cultures have lost their practical wisdom partly because of the way our students are educated and our educational institutions are organized, creating barriers of rank, tasks, and cognition between people. He says that the epistemic logic of decision-making resting on objective, impersonal **IS** are highly data dependent, but have little meaning for social discourse (Bernstein, 1983; Lanir 1993).

COMMUNICATION THEORY OF SOCIETY

Habermas (1979, 1987) is recognized the world over as one of the most important figures in modern sociology, particularly for his major contributions to the theory of communication. Some aspects of his theory are most relevant to **IS**. In the next section we introduce Habermas's communicative theory, by prefacing it with a more focussed explanation of instrumental, communicative, and emancipatory rationality.

Instrumental rationality is linked to principles of control and certainty. It uses the natural sciences as its model and rests on a number of assumptions that underlie its view of knowledge, human values and social enquiry. The scientific method believes in its supreme power to answer all significant questions, including the social. First, instrumental rationality operates on propositions which are empirically testable, such as mastery of the physical world. Second, knowledge, like scientific inquiry, is considered value free. Third, "hard" data is considered the focus of explanation and discovery. And fourth, not only is knowledge objectified, it is reduced to the mastery of technical decisions for ends already decided upon (Giroux, 1983). Instrumental rationality treats education as the means to a given end, and views teaching as a skilled craft akin to "techne" based on technical expertise. This leads to the idea that education can be improved by gaining a more complete mapping of the cause and effect relation in education, very similar to input-output systems, where the emphasis is on control and conformity through behavior modification and competency (Ewert, 1991).

On the other hand, communicative rationality has a deep-seated interest in understanding social interaction. Human beings are never seen as passive recipients of information. They are considered sensitive to the notion that through the use of language human beings constantly produce meanings as interpretations of the world in which they find themselves. In short, human beings (rather than nature) are seen as the ultimate authors of knowledge and reality (Giroux, 1983). Communicative rationality considers the aims of education as the criteria for the process of education as a social activity with social consequences, engaged in social reproduction. What control is possible is through the wise decision making of practitioners: justifying actions with reference to norms; acting prudently in situations of normative conflicts; and aiming towards consensus in case of differences (Ewert, 1991).

Lastly, emancipatory rationality attempts to locate meaning and action in a societal context, basing its principles on criticizing that which is restrictive and oppressive, while at the same time supporting action in the service of individual freedom and well-being. Emancipatory rationality augments interest in self-reflection, designed to create conditions in which non-alienating and non-exploitative relationships exist. Emancipatory interests are reflected in education in the drive to transcend, to grow, and to develop. The key element is the capacity of the student to be reflective and articulate (Ewert, 1991).

Rationality	**Knowledge**	**Medium**	**Methodology**
Technical	Instrumental	Work	Empirical
Practical	Communicative	Language	Interpretive (Hermeneutic)
Emancipatory	Reflection	Power	Critique

Figure 3. The Three Rationalities

The characteristics and attributes of the three rationalities described above are shown in Figure 3. Habermas calls them knowledge-constitutive interests because they guide and shape the way knowledge is constituted in human activities.

HABERMAS' COMMUNICATIVE THEORY & VALIDITY CLAIMS

The "communicative theory of society" developed by Habermas has a direct application to **IS**. According to Habermas, four different kinds of speech acts are related to four corresponding validity claims which are thematically stressed in discourse:

Truth: All participants contributing to the discourse must have equal opportunity to use truthfully grounded information and arguments, so that explanations and interpretations are properly aired and no particular view expressed is exempted from consideration and/or criticism.

Rightness: All participants should have an equal chance to command and oppose, permit and forbid information and arguments. This requirement ensures that participants raise fully appropriate arguments with respect to issues. To speak legitimately, in context, is the essence of this speech act.

Sincerity: The opportunity to express attitudes, feelings, and intentions regarding information must be available to all participants. Freedom from internal constraints must be maintained, by requiring all participants to be honest and sincere to themselves and their colleagues.

Comprehensibility: This requirement demands that all participants use communicative speech that is understandable to everybody. Confusion and chaos would be the net result, if nobody understood what a speaker was saying. Consequently, participants must take steps to ensure the speaker's meaning. These validity claims guarantees that a rationally grounded consensus can emerge from practical (or communicative) discourse.

TEACHING EXPERIMENTS

The various components of the **IS** described in this paper, from data gathering right through the knowledge and intelligence stage, were used in six undergraduate courses that included a heavy mathematical problem solving content. A limited attempt was made to introduce the use of knowledge-based expert systems and in applying artificial intelligence in team-based exercises. Class evaluations at the end of each semester revealed that about 80% of the students grasped the application of the mean/ends configuration, the concepts of rationality, and Habermas' communication theory and validity claims. Whilst only a couple of hours in each class were spent in introducing Aristotle's theory of "practical wisdom", their individual interests in this area were not recorded. Almost 95% of the students were good "number-crunchers" when confronted with back-of-the-chapter type problems. However, they were found to be average to poor performers at tackling "wicked" real-world problems. Very few students ever questioned the validity of the data or its sources, which came as a surprise. Students were generally very interested in the classification of information, such as suggestive, conceptual, deontic, predictive, decisive, systemic, etc. They were equally interested in different kinds of input-output models fueled by this information classification.

The general comments of students were: I now have greater understanding of, and confidence with, using **IS** for mathematical problem solving; the hierarchy of **IS** is important to me; I feel I understand the why, how, what, and when, of **IS**; I wonder why the components if **IS** were not explained to me before, in other courses; I enjoyed the exercises in artificial intelligence and knowledge-based expert system design; and the list goes on. There were very few negative comments, mostly reflecting the physical facilities and conveniences provided for the students. Overall, my colleagues and I were satisfied with the results we achieved in our classes, although we wanted to forge ahead with the plan developed in the theoretical framework.

REFLECTIONS AND ONGOING RESEARCH

We are cognizant that current reforms in mathematics education suggest that the learning of mathematics be very closely connected and blended with its applications (NCTM, 1989). More and more mathematics curricula provide opportunities for students at all levels to collect their own data sets, analyze them, and make judgements about their appropriateness before the analysis phase. In some cases, this activity is connected to students investigating their own neighborhoods for gathering and selecting socio-economic data for myriads of real-world neighborhood problems, e.g., crime, accidents, mortality rates etc., (Khisty, 1996). It is our premise that if mathematics is to be truly connected to resolving socio-economic problems, then students need to be more than just number-crunchers. Critical to the understanding and practice of mathematics is the ability to develop practical wisdom, putting communicative action into operation, and appreciating the efficacy of emancipatory interests.

The theoretical framework described in this report has now been partially put into classroom practice and the results are encouraging to both teachers and students. There is no question that further work needs to be done, and the tasks we want to take up next are briefly described below:

- How should students be taught the hierarchy of **IS** so that they begin to appreciate the authenticity of data sets through objective and subjective testing?
- What would be the most effective methods of introducing the ideas of practical wisdom while working with real-world problems?
- What are the mechanisms of individual and collective cognition through which students acquire a level of theoretical and practical competence in handling real-life problems where mathematical and socio-economic variables are being considered?
- In addition to deductive and inductive reasoning, how can students be introduced to abductive inferencing for solving "wicked" problems (Khisty & Khisty, 1992) they encounter on a daily basis?
- How should instructors go about preparing at least a preliminary prospectus for an "instructors manual" as suggested by Machado?
- How can this research be extended to high school and freshmen undergraduate students?
- How should we prepare teachers to teach mathematics courses using **IS** so that they embody the socio-economic complexities of the world.

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COLLABORATIVE VERSUS INDIVIDUAL PROBLEM SOLVING: ENTERING ANOTHER'S UNIVERSE OF THOUGHT

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Abstract. *This paper presents findings pertinent to the growing literature on the role of collaborative problem solving in the learning of school mathematics. The mathematical discourse of two, high-ability, 13-year-old boys working together on a sequence of problems that were novel to them is examined, with a view to observing the difficulties involved in entering another's universe of thought. Five different types of interaction are identified in the analysis of the conditions under which their partner-directed talk was/was not beneficial. The individual problem-solving work, which occurred just after their joint work, is compared with what they were able to produce collaboratively. Findings suggest that the brief moments during which one of the interlocutors was able to enter the universe of thought of the other played a fundamental role.*

Much current research in mathematics learning is focusing on the facilitating role played by collaborative settings. For example, Leikin and Zaslavsky (1997) have shown how small group cooperative settings lead to higher engagement levels, improved attitudes, and increased mathematical communication among low-achieving ninth grade students. Hershkowitz and Schwarz (1997) have demonstrated how four ninth grade students interacted to produce and check hypotheses, and how they learned to be critical and reflect on their own and others' problem solution processes. And Teasley (1992), who compared the quality of work of fourth graders in four different experimental settings involving a Logo problem-solving task, has found that talking is of significant benefit to the learning process, and that these benefits are more pronounced when that talk is directed to a partner. But the research on learning from interaction raises many questions. Is all partner-directed talk beneficial? Is it beneficial to the speaker, to the listener, or to both? Under what conditions is it beneficial; what makes it so? And what, precisely, do we mean by beneficial?

Pirie (1996) has drawn out some of the complexities involved for teachers when they really *listen* to the oral communication of students. Certainly, similar difficulties arise when students attempt to listen to each other as they are doing mathematical work in small group settings. Trognon (1993) has pointed out that interlocutors can be listening to each other, but it may be the case that they do not understand each other. The moments when one truly enters the "universe of thought" (p. 341) of the other may be quite brief, but they play a fundamental role.

This paper examines the mathematical discourse of two, high ability, 13-year-old boys working together on a sequence of problems that were novel to them. In observing the difficulties involved in entering another's universe of thought and in analyzing those moments when it did seem to occur, we identify five different types of interaction engaged in by the boys. The paper follows with a brief description of the boys' individual problem-solving work, which took place the day after the joint activity, and discusses how it relates to what they were able to produce collaboratively and how it reflects the learning that occurred/did not occur during the joint work.

BACKGROUND

History: The two boys featured in this paper are in their first year of high school. They have just completed a seven-week introduction to algebra based on an object-oriented, functional approach, which is aimed at giving meaning to the symbols and transformations of algebra by means of graphical representations and operations with these representations (see Kieran, 1994). Much of the seven-week sequence had involved pairwise work with activity sheets, interspersed with classroom instruction by the teacher and whole-class discussions. The content that was emphasized in these activity sheets was primarily situations involving linear functions, although some experience with quadratic and cubic functions was included. Several days of class work were spent at the computer, where students in groups of two were able to explore the role of the parameters in graphical and symbolic representations, and their relation to the problem situations. Thus, collaborative interaction was not new to these two boys, who often worked together as a pair in their math class.

Research Program: The analysis presented in this paper is part of a larger study in a research program focusing on alternate approaches to the introduction of algebra in environments that include a computer component. The larger study had three aims: a) to see whether the object-oriented introduction to algebra equips the students with the thinking tools needed to explore the graphical, algebra-symbolic, and verbal representations of functions that are unlike those they have studied in class; b) to explore the role played by pairwise interactions in problem-solving situations; and c) to investigate the extent to which students freely choose to use the computer as a problem-solving tool. The part that is presented in this paper is related primarily to aim (b), which focused on the discourse and problem solving of pairs of students. Thus, evidence was collected that would permit us to answer questions regarding the nature of their collaborations and the relation between their collaborative and individual work.

Methodology: One pair of students was observed at a time, but this observation was from a distance. After a couple of warm-up questions, the pair was given one set of shared activity sheets containing eight questions--some with sub-questions; they were asked to work together in the solving of the given problems, taking as much time as they needed (the tasks were designed so as to take about 45 minutes to an hour to complete). There was to be no adult intervention and all their interactions were to be videotaped. A computer was beside their work table available for their use, should they so decide. The next day, each member of the pair returned to do individual problem solving on tasks that were analogous to those worked on jointly the day before. This part of the methodology was meant to help us capture what they may have learned as individuals during their collaboration. This individual work was also videotaped.

THE JOINT PROBLEM SOLVING AND TYPES OF INTERACTION

We now present excerpts from the verbatim transcripts of the videotapes of one pair of boys, Nick and Zak, that illustrate the ways in which they worked together. The set of tasks was called HEADWINDS AND TAILWINDS.[1] The first situation was the following:

> *Glen and Matt own and fly experimental small airplanes. With no wind, Glen's plane can fly at 50 kilometers per hour (kph) and Matt's plane at 60 kilometers per hour. Glen and Matt are planning a trip to an airport 150 kilometers away from their home. The time for the trip will depend on the wind speed. A headwind will make the trip longer; a tailwind will make the trip shorter. The graphs in Figure 1 show the relationship between wind speed and time for the trip for each flier and for different headwind or tailwind conditions.*

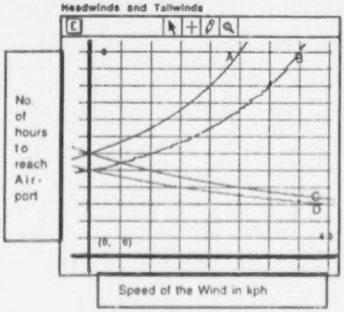

FIGURE 1

Zak generated his interpretation of the headwind and tailwind graphs immediately upon seeing the graphs. He visualized the graphs globally, seeing the upper graphs as headwinds pushing against the plane and the lower graphs as tailwinds going with the plane. Later comments tended to confirm this interpretation. Nick, on the other hand, had a far more analytical, pointwise, numeric approach, trying to draw conclusions from the particular values the functions were taking at specific points.

Pragmatic Interaction: After a few introductory questions (1a - 1h) whose purpose was to ensure they understood basically how to read the given graphs, they started working on the following question:

> 2. *"Try to match each of the graphs in Figure 1 with one of the following situations: Glen and Tailwind, Glen and Headwind, Matt and Tailwind, Matt and Headwind."*

The following excerpt reflects what happened after they decided that "Glen and Tailwind" was Graph C:

170 Z: Glen and headwind is figure A.
171 N: Wait. You sure? Wait.
172 Z: Yeah, cause you got A B C or D.
173 N: Just a sec. Glen's is 50, so if you have (hesitates). No no no no, Glen's at B cause it goes
174 Z: Well Glen is
175 N: I say Matt's C, but I'm not sure, cause look
176 Z: Oh yeah. No no no no, Matt's B (points at Graph B). I thought we were doing that, okay.
177 N: Matt's D.
178 Z: Glen and headwind is A (pointing at Graph A).
179 N: What?

180 Z: This is Glen's graph (tracing with his finger Graph A from right to left, and then Graph C from left to right, in one continuous motion). C and A is Glen's graph.
181 N: Yeah yeah yeah, you're right, sorry.
182 Z: Matt at tailwind is D.
183 N: (not understandable)
184 Z: Yeah, and Matt at headwind is B.
185 N: I still think, it doesn't matter.

This excerpt illustrates that two students may work together on a task, make reasonable progress toward a solution while talking to each other, probably even listening to each other, but without any proper interaction to speak of: Their interaction does not reflect their thinking. Each one appears to be contemplating the problem for himself, in his own way, within his own universe; when trying to convince the other, however, they do not use reasoning but simple statements, forcefully brought forward. We propose to call this **pragmatic interaction**. It may appear that Zak, using his physical universe of thought, dominates here while Nick is lost. However, as is shown later by Nick's individual work, he is perfectly able to produce convincing arguments of the kind that are needed to answer Question 2; only, his arguments are of a different type, grounded in a different universe of thought. We submit that the superficiality of their interaction in this excerpt is due to the fact that their universes of thought are incompatible. And Nick decides, in pragmatic fashion, not to pursue the issue.

Homogeneous Interaction: When two people think alike, in overlapping parts of their respective universes, they can often work together quite well. We have elected to call **homogeneous interaction** the type of interaction that occurs in such a situation. This type of interaction did transpire between Nick and Zak but, for lack of space, we cannot illustrate it here.

Pseudo-Interaction: Sometimes, the degree of interaction between Nick and Zak was even lower than that seen in the pragmatic interaction episode; Question 7a asked them to try to come up with the expression that would produce Graph B. The following excerpt presents part of their talk during that episode:

407 Z: Try to come up with the expression that will produce Graph B.
408 N: Okay.
409 Z: What's Graph B? Oh this one (referring back to Figure 1).
410 N: Hmm hmm.
411 Z: So you got to do
412 N: It's 2.5 (pointing to (0 2.5) of Figure 1), so you have to something
413 Z: It's 2.5? No, wait, no no no, okay, the headwind is is, I mean, it's rising (following Graph B with his pen).
414 N: 10 divided by, no, 10 divided by, no, 10 divided by, no.
415 Z: In an hour, in one hour, where's an hour, it goes up (here he traces a portion of Graph B, between y=3 and y=4).
416 N: 5 divided by 2 minus x.
417 Z: 10 to 25.
418 N: 5 divided by 2 minus x.
419 Z: 23. What? (was not listening to what N was saying).

During this episode, the two boys worked on the same problem, finding an expression that would represent Graph B, each using his own approach. A superficial look at the protocol appears to show that they took turns talking to each other, but a closer examination reveals that they were hardly listening to each other; rather, each was working by himself on a separate line of thought, within his own universe and talking more to himself than to the other: Nick considered the numerical question of how to make up an expression that would yield the value 2.5 at x=0, whereas Zak thought about the modeling question of how a headwind could be taken into account in an expression. We propose to call this a **pseudo-interaction**: There was no interaction going on at all--it only seemed as if they were talking to each other.

Inhomogeneous Interaction: Not always were Nick and Zak so far apart, cognitively. The question immediately following 7a was the analogous question of finding an expression representing Graph D, a question for which they did interact, and in a very deep sense. The following excerpt starts with Nick asking about the origin of some of the data they had used earlier, namely the numbers 150 and 50 which appear in the expression for Graph A. The reason Nick now wants to understand the origin of these numbers is that he hopes to solve the task at hand, building an expression for D, by analogy.

438 N: But where did they get their 150 and 50?
439 Z: They get the 150 by the time it takes, it's 150 km away from the airport.
440 N: Oh yeah?
441 Z: Yes.
442 N: Okay, so then this will be
443 Z: 150 divided by
444 N: 150 divided by (said at the same time as line 443) (the two boys then hesitate).
445 N: What gives a 2.5 answer?
446 Z: 2.5 is 5 divided by 2 plus x.
447 N: What?
448 Z: 5 divided, no you don't even need the 5 divided by, you have 2.5, 2.5 plus x (writes 2.5+x).
449 N: No no no, you have to (do different?) I think, cause you have to put the whole thing (pointing to the expression $\frac{150}{50 - x}$ in question 6a).
450 N: (after a short pause) It's, hmm it's okay, this is the expression.
451 Z: Oh yeah.
452 N: 150 over 60 plus tw..., no plus x. Okay? I say this is my expression there (writes $\frac{150}{60 + x}$).
453 Z: 60 + x, is this, no, Graph B is
454 N: Wait (a second?).
455 Z: Okay it sounds
456 N: The 150 is away from the hmm
457 Z: Is how far
458 N: Airport?
459 Z: The speed of the wind.
460 N: Okay, that's good for me.
461 Z: How far is for km, it's 150 km now (said at the same time as line 460).
462 N: But which one is that, is that B?
463 Z: It's 150 km away.
464 N: Yeah okay.
465 Z: 150 km divided by 60 + x, **yes** (emphatic), **that's it!**

During this episode, the boys efficiently collaborate to find a solution, each providing what he is best at: Nick the numerical details of the model and Zak the general structure and its interpretation. Each makes an honest (though not always successful) attempt at understanding the other's contribution. Such a strong interaction, where each participant presents his own thinking in spite of being conscious that the other finds it difficult to follow, requires a considerable mental effort and willingness to learn rather than just get done with the task. Here Nick and Zak proved that they can, and sometimes will, make such an effort. We propose to call this **inhomogeneous interaction**, considering the fact that the basis for the interaction is constituted by differences in the students' universes.

Anti-Interaction: It also happened that the boys refused to interact, such as when Zak threw a "Be quiet" at Nick, clearly pointing to his need to be left alone, to be given some time to quietly think things through for himself, within his universe of thought without any intervention from outside, without any collaboration, group work etc. We propose to call this **anti-interaction** and we submit that most people (not only children), when learning mathematics, need some extended periods of quiet concentrated thought by themselves.

THE INDIVIDUAL PROBLEM SOLVING

From the few excerpts above, it would seem that Zak's explanations as to why 150/(50-x) should be the expression for Graph A "took" with Nick. Zak had been able to state how this expression made explicit the mathematical givens of the problem and the relation between those givens. And the set of operations in the expression did, after all, represent exactly the point-wise calculations that Nick carried out when asked later for the sketch of a graph. But an examination of Nick's individual work suggests that the expression that made sense in Zak's universe of thought was not appropriated by Nick, even though the details of the expression fit with aspects of Nick's universe.

The set of tasks on which Nick and Zak worked individually was called RIVER CURRENTS. It consisted of four questions that were for the most part analogous to the kinds of questions they had worked on jointly. The initial situation was:

> *Susan is training this spring for the long-distance marathon swim competition that will take place at the end of the summer. Her training consists of a daily swim up the Richelieu River, going against the current, for a distance of 20 kilometers. When there is no current, Susan swims at a speed of 4 kilometers per hour (kph). The speed of the current can vary from one day to the next. The time that Susan takes to do her swim of 20 kilometers depends on the speed of the current.*

When asked Question 1b, "Which one of the graphs accurately depicts the number of hours Susan could take when she is swimming against the current?", Nick chose the correct graph. But for the next two questions, which asked the reason for choosing that particular graph, and the expression that goes with that graph, he responded according to a linear interpretation, with the expression: 5 + 2.5x. By taking one point off the y-axis, along with the y-intercept, he constructed a "slope-y intercept" expression for the curve. The interactive work that had led to correct expressions for the Headwinds and Tailwinds graphs, and which might have suggested that both boys had "got it," had not been sufficient for Nick to enter Zak's universe of thought and thereby restructure his own. This contrasts with Zak's individual work. For Question 1b, Zak produced the expression 20 / (4 - x), which reflects the responses that the two boys produced in their joint work on analogous Headwinds and Tailwinds questions. Thus, it appears that Nick never really understood the appropriateness of the expressions they had produced jointly and relied instead in his individual work on the linear form that he had learned in class.

DISCUSSION AND CONCLUSIONS

Returning now to the questions that were posed earlier regarding the benefits of partner-directed talk, we need first to address the issue of what we mean by *beneficial*. What we mean concerns whether or not that which the boys were able to do at an interpersonal level becomes interiorized for each of them. The fact that a key discursive moment for Zak had taken place was reflected in line 465. It was at that moment that Zak, in response to the many times that Nick had asked, "Where does the 150 come from; where does the 50 come from?", *appropriated intrasubjectively* (see Trognon, 1993) elements from Nick's discourse, that is, elements related to a less global, more detailed universe of thought. The irony is that the explanation Zak offered was not internalized by Nick, even though Nick's questioning had provoked it.

Thus, we have a case here where the partner-directed talk was more beneficial for the speaker Zak than for the listener Nick. But the sequence that led to this result was not immediate; it was spread over a five-minute interaction: From line 369 in the transcript, when both boys decided by elimination that 150/(50-x) was the best choice for Graph A of Figure 1, followed by Nick's request in line 370 that Zak "now explain it," to Zak's statement in line 385 "But I can't explain it though," to Nick's continued request in line 398 "Why is this 50-x since you understand it so good?" Line 403 has Zak offering that "if you're minusing x, it's the headwind," and that "50 minus let's say 10 is 40, and 150 by 40" (line 405); but again in line 432, Nick commented, "I don't know, I don't know where the 150 and the 50 come from." In the interaction documented above that includes lines 438-465, we note that, with respect to Nick's questioning, Zak specifically mentioned that the 150 was due to "it's 150 km away from the airport," but more importantly that Zak seemed to be trying to work something out for himself, that he was talking to himself, as much or even more than he was talking to his partner. In fact, when the partner was prepared to move on to the next question (line 460), Zak insisted on staying with the problem at hand. We submit that it was this period of **internal talk**, which had occurred within the context of the partner-directed talk, that was crucial for Zak. The period of internal talk permitted Zak, having briefly entered Nick's universe of thought to explain the details of the expressions, to relate those details back to his own universe of thought. This empirically-supported theoretical perspective, inspired by Trognon (1993), can be used to explain why Webb (1991) and others have been able to claim that giving help is often of more benefit to the help-giver than to the help-receiver.

We have identified five possible types of (lack of) interaction during the work of a pair of students, and related these types of interaction to differences in the students' universes of thought: Anti-interaction, Pseudo-interaction, Pragmatic interaction, Homogeneous interaction, Inhomogeneous interaction. We do not want to suggest that students should not interact, that work in pairs (or groups) is inefficient in mathematics learning. We do, however, want to point out that work in pairs (and groups) is not unproblematic, and that students should be given ample time to work and to think on their own. It was during the period of **inhomogeneous interaction** that Zak was able to reach a peak with respect to structuring his cognition, but this movement was due in large measure to what we have called, **internal talk**, which appeared to be occurring at the same time. For Nick to have made a similar kind of progress, we can only conjecture that interaction of a different nature might have been necessary--a type of interaction that did not occur between these two boys--an interaction that would have provoked him to engage in internal talk relating the two universes of thought. For even when Nick was given the time to work things out on his own, as in the individual work, this individual work did not reflect the understandings that had been suggested by their joint work. We can only conclude that Nick remained throughout in his own universe of thought.

ACKNOWLEDGMENTS

[1] The questions developed for this sequence were inspired by a task created by Fey & Heid (1991).

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INCREASING TEACHERS' AWARENESS OF STUDENTS' CONCEPTIONS OF OPERATIONS WITH RATIONAL NUMBERS[1]

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ABSTRACT: *This paper describes prospective and inservice teachers' knowledge of children's common difficulties in calculating multiplication and division expressions involving rational numbers and their possible sources. Our data show that prospective teachers who specialized in mathematics knew more about students' common incorrect responses than those who specialized in teaching other topics. Inservice teachers that experienced teaching rational numbers were more knowledgeable about students' common mistakes than those who had practice in teaching other mathematical topics. A major finding is that participation in a course on students' conceptions of rational numbers dramatically increased both prospective and inservice teachers' awareness of students' incorrect responses and their possible sources.*

Educators generally agree that learning occurs by challenging (or building upon) existing conceptions. Consequently, calls for reform in mathematics education emphasize that knowledge of students' common conceptions and misconceptions about the subject matter is essential for teaching (e.g., Australian Education Council, 1991; National Council of Teachers of Mathematics, 1989; 1991). Recent studies have, however, reported that prospective teachers' ability to analyze the reasoning behind students' responses was poor (e.g., Even & Tirosh, 1995). Thus it seems that a major goal of teacher education programs should be to promote prospective and inservice teachers' knowledge of children's ways of thinking about the mathematic topics they are to teach.

In most countries, a substantial part of the curricula of elementary schools is devoted to rational numbers. There is a considerable body of research reporting that students experience difficulties with the operations of multiplication and division with rational numbers (e.g., Barash & Klein, 1996; Carpenter, Lindquist, Brow, Kouba, Silver, & Swafford, 1988; Hart, 1981; Fendel, 1987). Three main sources of these difficulties are often proffered: algorithmically-based mistakes (various "bugs" in computing the expressions, e.g., $9\cdot\frac{3}{10}=\frac{9\cdot 3}{10\cdot 3}$), intuitively-based mistakes (overgeneralizing properties of operations with natural numbers to fractions, such as the divisor must be smaller than the dividend) and formally-based mistakes (incorrect performance due to limited conceptions of the notion of fraction and the properties of operations with fraction, e.g., assuming that division is commutative and therefore $1:\frac{1}{2}=\frac{1}{2}:1=\frac{1}{2}$).

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This paper is a result of a project whose main aims were: (1) to describe prospective and inservice elementary school teachers' Subject-Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) of rational numbers; and (2) to develop teaching strategies for enhancing teachers' SMK and PCK of rational numbers and evaluate their impact. In a previous paper, we described prospective and inservice teachers' PCK of multiplication and division word problems (Klein & Tirosh, 1997). In this paper we describe prospective and inservice teachers PCK of common children's difficulties in calculating multiplication and division expressions involving rational numbers and of their possible sources. We also briefly describe a course aimed at enhancing teachers' SMK and PCK of rational numbers, and report on the impact of this course on prospective and inservice teachers' PCK of this topic. Our main research questions are:

(1) Are prospective and inservice teachers aware of common difficulties that children experience in calculating multiplication and division expressions involving rational numbers? To what do they attribute them?

(2) What are the effects of teaching experience on teachers' awareness of students' difficulties in calculating multiplication and division expressions with rational numbers and their possible sources?

(3) What is the impact of participation in a course that focuses on students' ways of thinking about rational numbers on prospective and inservice teachers' PCK of this topic?

Methodology

Subjects. Ninety seven Prospective Teachers (PT) and 118 Inservice Teachers (IT) participated in this study. Thirty-seven of the prospective teachers were in their first year, 30 were in their second year and 30 were in their third year in a four-years elementary teacher education program at an Israeli State Teachers' College. Prospective teachers in this college choose a domain of specialization. Only those who had participated in a relatively extended mathematical program in high-school could choose mathematics as their domain of specialization, and their program included substantially more mathematics and mathematics method courses than the program of those who specialized in other domains. The second year prospective teachers who participated in this study specialized in mathematics.

The participating inservice teachers took a special two-year program aimed at creating a community of leading elementary school mathematics teachers in Israel. This Expert Teachers Program (ETP) included mathematical courses, pedagogical courses, and work-field. Sixty four of these inservice teachers were in their first year of the program and 54 were in their second year. Most of the participating inservice teachers had experience in teaching rational numbers (41 students in the first year and 41 in the 2nd year). The others were teachers who taught mathematics in classes where rational numbers were not included as a main topic in the curriculum.

We shall report on the similarities and differences between SMK and PCK of non mathematics majoring prospective teachers, mathematics majoring prospective teachers, and inservice teachers.

Instruments. Two main instruments were used in this study: 1. A Diagnostic Questionnaire (DQ) 2. a course on Students' ways of Thinking About Rational numbers (STAR).

1. The DQ included the following problem: *Following are nine expressions: (a) calculate each of these expressions, (b) list common mistakes students in seventh grade may make, after finishing their studies of fractions, and (c) describe possible sources for each of these mistakes.*

$$9\cdot\frac{3}{10} \qquad \frac{3}{4}\cdot\frac{1}{2} \qquad 320\cdot\frac{1}{3} \qquad \frac{1}{4}:4 \qquad \frac{1}{4}:\frac{3}{5} \qquad 5:15 \qquad 4:\frac{1}{4} \qquad 15:5 \qquad 320:\frac{1}{3}$$

Research in mathematics education reports on common difficulties that students experience with eight of these nine expressions (see Table 1 columns 1& 2). One of the expressions (15:5) is usually computed correctly by students. Including this item enabled us to examine the participants' ability to differentiate between expressions that are often incorrectly calculated and those that are usually correctly solved. The inclusion of two pairs of expressions (15:5 and 5:15, $\frac{1}{4}:4$ and $4:\frac{1}{4}$) yields a more comprehensive picture of teachers' PCK.

2. The course (STAR). This 30 hours (one semester, two weekly hours) course was aimed at developing teachers' understanding of mathematical concepts and structures related to rational numbers. Another major aim was to enhance the participants' knowledge of children's ways of thinking about rational numbers. Use was made of relevant research findings on children's, prospective and inservice teachers' conceptions of rational numbers. The participants' ways of thinking about and with rational numbers were used as a springboard to discuss various issues (i.e., the mathematical reasons behind the relatively complicated definition of addition of fractions, alternative definitions of division, etc.)

Procedure: The DQ was administered to all subjects in two sessions of 90 minutes each, during a mathematics lesson. The second year prospective teachers and the inservice teachers in their second year of study in the ETP program participated in the course on students' ways of thinking about rational numbers. The DQ questionnaire was administered to the second year prospective teachers before the course and again about a month after it. The inservice teachers enrolled in the second year of the ETP program answered the DQ about a month after participating in the course. It is noteworthy that while prospective teachers who responded to the DQ before and after instruction were the same students, the inservice teachers before and after instruction were different groups (inservice teachers in the first and second years of the program, respectively). Yet, these two groups of inservice teachers were very similar in terms of their mathematical background and teaching experience, and therefore it was possible to assess the effects of participation in the course by comparing the responses of these two groups to the DQ questionnaire.

Results

Prospective and Inservice Teachers' SMK: The analysis of teachers' PCK should take account of teachers' own solutions. Therefore, we first analyze the subjects' responses to the multiplication and division examples. Our data show that the vast majority of the prospective and inservice teachers correctly calculated all nine multiplication and division expressions. The percentages of correct answers to each item did not fall below 85%. Still, several responded incorrectly to some of these items. For instance, 16 subjects (7%) gave an incorrect response to $\frac{1}{4}:4$ usually getting a quotient of 1 instead of $\frac{1}{16}$ (they wrote $\frac{1}{4}:4=\frac{1}{4}\cdot 4=1$). It is noteworthy that some participating teachers who incorrectly calculated the expressions considered the correct answer to be incorrect. Moreover, for the expression $\frac{1}{4}:\frac{3}{5}$, 9% of the teachers who answered correctly, considered the correct expression $\frac{1:3}{4:5}$ to be incorrect.

Prospective and Inservice Teachers' PCK Parts (b) and (c) of the problem probed teachers' knowledge of children's ways of thinking about multiplication and division expressions involving rational numbers. The analysis of PCK to each item relates only to the subjects who correctly answered the item, as they were the vast majority of the participants.

Teachers' Knowledge of Common Students' Incorrect Responses: Table 1 describes the common incorrect responses that teachers who did not (or did not yet) participate in STAR listed to eight of the nine expressions included in the DQ.

Table 1- *Distribution (in %) of teachers' knowledge of common students' incorrect responses to computation expressions*

Expressions	common mistakes	PT:			IT:	
year		1st	3rd	2nd	1st	
multiplication		not majoring		majoring	not teaching	teaching
$9\cdot\frac{3}{10}$	$\frac{9\cdot 3}{9\cdot 10};\frac{3}{9\cdot 10};\frac{?}{9\cdot 10}$	47	63	86	38	59
$\frac{3}{4}\cdot\frac{1}{2}$	$\frac{3}{4}\cdot\frac{2}{1}$; common denominator	43	46	41	10	32
$320\cdot\frac{1}{3}$	$\frac{320\cdot 1}{320\cdot 3};\frac{1}{320\cdot 3}$	33	42	65	38	44
division						
$\frac{1}{4}:4$	$1;\frac{1}{4}\cdot 4;4:\frac{1}{4};4\cdot 4;4\cdot\frac{1}{4}$; impossible	52	56	70	67	92
$\frac{1}{4}:\frac{3}{5}$	$\frac{1}{4}\cdot\frac{3}{5};\frac{4}{1}\cdot\frac{3}{5};\frac{4}{1}\cdot\frac{5}{3}$; impossible	45	37	45	38	71
5:15	15:5 ; impossible	23	28	50	43	55
$4:\frac{1}{4}$	$1;\frac{1}{4}\cdot\frac{1}{4};4\cdot\frac{1}{4}$	34	41	59	50	65
$320:\frac{1}{3}$	$320:3;\frac{1}{320}\cdot\frac{1}{3};320\cdot\frac{1}{3}$	15	27	64	43	58
Average		36	43	60	40	59

As mentioned before, students usually correctly respond to the item 15:5. The majority of the participating teachers mentioned no mistakes to this expression.

Table 1 shows that prospective teachers majoring in mathematics mentioned more common difficulties that students tend to experience than those who did not major in mathematics (60% compared to 40%). Inservice teachers who practice teaching rational numbers were more aware of students' common difficulties than inservice teachers who did not practice such teaching (59% compared to 40%). Table 1 also shows that the averages of common incorrect responses mentioned by prospective teachers who were not majoring in mathematics and by inservice teachers who did not teach this topic were very similar (around 40%). The averages of prospective teachers majoring in mathematics and those of inservice teachers who teach rational numbers were also similar (around 60%).

Table 2 shows that teachers who participated in STAR had more profound knowledge about common incorrect responses to multiplication and division expressions than those who did not. The data show that all three groups of teachers (prospective teachers, inservice teachers with no experience in teaching rational numbers and inservice teachers who practice teaching rational numbers) gained from participation in STAR. Interestingly, no substantial differences in ability to list students' common mistakes were observed between these three groups of teachers after participation in the course.

Table 2- *Distribution (in %) of teachers' knowledge of common students' incorrect responses to computation expressions before and after participation in STAR*

Expression	PT:		IT: not teaching		teaching	
year	2nd		1st	2nd	1st	2nd
participation in STAR	-	+	-	+	-	+
multiplication						
$9 \cdot \frac{3}{10}$	86	90	38	64	59	49
$\frac{3}{4} \cdot \frac{1}{2}$	41	45	10	27	32	35
$320 \cdot \frac{1}{3}$	65	70	38	45	44	37
division						
$\frac{1}{4} : 4$	70	97	67	100	92	83
$\frac{1}{4} : \frac{3}{5}$	45	51	38	45	71	59
$5 : 15$	50	79	43	82	55	88
$4 : \frac{1}{4}$	59	79	50	80	65	79
$320 : \frac{1}{3}$	64	77	43	80	58	79
Average	60	72	40	65	59	63

Teachers' Knowledge of Possible Sources of Students' Incorrect Responses: In part (c) the subjects were asked to describe possible sources for each of the mistakes they listed in response to part (b). As could be expected, most of the sources mentioned by the teachers to all these computational expressions were algorithmically-based (e.g., "not writing the integer in fraction form", "inverting both the dividend and the divisor"). Very few participants (mainly the prospective teachers who specialized in mathematics, and the inservice teachers that practice teaching rational numbers) related to both algorithmic and intuitive sources of incorrect responses (e.g., "students will think that it is impossible to divide a small number by a larger one").

Table 3 shows that before participation in STAR, most prospective and inservice teachers were unable to mention possible sources of students' incorrect responses (many participants repeated the mistake itself, or didn't respond to this item). The lowest percentage of sources of students' mistakes (17%), was given by non mathematics majoring prospective teachers (first and third years). Somewhat higher percentages were observed among non teaching inservice teachers (33%). The highest percentages were given by prospective teachers majoring in mathematics and inservice teachers who have experience teaching rational numbers, but even these percentages were rather low (48% and 41% respectively).

Table 3- *Distribution (in %) of teachers' responses to sources of students' incorrect responses to computation expressions*

	PT			IT	
Expressions	not majoring		majoring	not teaching	teaching
year	1st	3rd	2nd	1st	
multiplication					
$9\cdot\frac{3}{10}$	9	18	73	29	38
$\frac{3}{4}\cdot\frac{1}{2}$	35	8	30	13	22
$320\cdot\frac{1}{3}$	10	19	49	24	44
division					
$\frac{1}{4}:4$	28	32	57	28	74
$\frac{1}{4}:\frac{3}{5}$	21	11	37	38	34
5:15	13	28	40	43	52
$4:\frac{1}{4}$	11	10	52	40	35
$320:\frac{1}{3}$	6	12	46	48	32
Average	17	17	48	33	41

Table 4 shows that participation in STAR contributes to teachers' awareness of possible sources of students' common mistakes to multiplication and division expressions. Participants in the course typically mentioned several possible sources of incorrect responses (e.g., algorithmically, intuitively and formally-based sources).

Table 4- *Distribution (in %) of teachers' responses to sources of students' incorrect responses to computation expressions before and after* STAR

Expression	PT:		IT: not teaching		teaching	
year	2nd		1st	2nd	1st	2nd
participation in STAR	-	+	-	+	-	+
multiplication						
$9\cdot\frac{3}{10}$	73	82	29	55	38	43
$\frac{3}{4}\cdot\frac{1}{2}$	30	48	13	40	22	44
$320\cdot\frac{1}{3}$	49	83	24	27	44	29
division						
$\frac{1}{4}:4$	57	93	28	91	74	72
$\frac{1}{4}:\frac{3}{5}$	37	54	38	72	34	59
$5:15$	40	94	43	55	52	82
$4:\frac{1}{4}$	52	62	40	80	35	53
$320:\frac{1}{3}$	46	79	48	100	32	47
Average	48	74	33	64	41	53

Conclusions

In this paper, we describe our findings regarding prospective and inservice teachers' PCK of children's ways of thinking about multiplication and division expressions involving rational numbers. Our data show that when exploring this aspect of PCK, it is crucial to distinguish between prospective teachers specializing in mathematics and those who major in other topics. The substantial differences in the PCK of the prospective teachers in these two groups of teachers are of special importance as all of them are intended to teach mathematics in elementary schools. The sources of these substantial differences and their implications on teacher education programs should further be explored.

We were expecting that inservice teachers' PCK of students' conceptions of multiplication and division expressions will be more elaborated than that of the prospective teachers. The data, however, provides a far more complicated picture. Teachers who teach this topic in schools indeed exhibited some knowledge about students' ways of thinking about multiplication and division expressions (similar to that of the prospective teachers who major in mathematics). But, the PCK of teachers who do not directly teach this topic is not elaborated, and remarkably similar to that of the prospective teachers who do not specialize in mathematics. Thus, our findings clearly indicate that teaching mathematics, by itself, does not necessarily enhance teachers' knowledge of students common ways of thinking. Furthermore, it seems that teachers do not develop by themselves general categorization of possible sources of students' difficulties, and they tend to resort mainly to one such source (algorithmic, in this case). We suggest that a general, theoretical framework related to cognitive processes and sources of misconceptions could support teachers in their attempts to foresee, interpret, explain and make sense of students' ways of thinking. Such models should be used only as a first, theoretically-based approximation that can assist teachers (and researchers) in their attempts to analyze the specific thinking processes that the child uses when approaching a specific task.

Our data reveal that participation in a special course on students' conceptions of rational numbers could enhance both prospective and inservice teachers' PCK of students' conceptions of multiplication and division expressions. Participation in the STAR course increased teachers' knowledge of common incorrect responses and provided them with a wider perspective regarding possible sources of such reactions. Yet, the impact of such knowledge on the actual teaching of rational numbers has still to be examined.

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INFLEXIBILITY IN TEACHERS' RATIO CONCEPTIONS

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Elementary school teachers, participating in a sequence of workshops on ratio and proportion, exhibited inflexibility in their conceptions. The teachers were able to perform conventional ratio and proportion tasks, but showed lack of flexible understanding of the concepts. Specifically, they could figure out ratios and perform ratio-division tasks, but failed to see the different appearances of the intensive ratio. The teachers also failed to realize that a modified additive strategy could be used to create equivalent ratios. This might be an over-generalization of the knowledge that an additive strategy is a wrong method in comparing ratios.

In a class of 28 students, the ratio between the number of girls and the number of boys is 4:3. How many girls are there?

This type of problem is cited by researchers and by the Third International Math and Science Study (TIMSS, 1997), an international research on math achievement, as an example of a difficult problem. The problem is often used in checking children's knowledge of ratio in tests that involve ratio and proportion concepts, and its low success rate serves to make a point that these concepts are difficult. Numerous articles have dealt with the identification and explanation of ratio and proportion difficulties. Some of them are: Noelting (1980a, 1980b), Tournaire & Pulos (1985), Hart (1981), Vergnaud (1988), Lesh, Post, Behr (1988), Confrey and Scarano (1995), to mention just a few of the contributors.

This study describes teachers' conceptions of some of the aspects of the ratio concept. The research is a part of an in-service teacher education project ("Tomorrow 98" in Upper Galilee), aimed at improving elementary school mathematics education in the spirit of the CGI project (1996). In this part we analyze the work of a group of 13 teachers, who are at least three years in the project, and have participated in a sequence of workshops (seven two-hour meetings) on ratio and proportion.

The Workshops: goals and structure

The goals of the workshops are to make teachers aware of children's difficulties in developing ratio and proportion concepts, improve teachers' own understanding of these concepts, and discuss and develop ideas about teaching them.

The teachers are introduced to children's understanding of ratio and proportion through Noelting's tasks. Noelting (1980a) describes steps in children's development of the concept of ratio and proportion, and suggests a mixture (of water and orange-juice glasses) task to diagnose the child's developmental level with regard to these concepts.

The teachers are given examples of children's answers at the different levels described by Noelting, and each teacher is asked to perform Noelting's test in a sixth grade class. In following meetings children's answers are discussed, additional issues are brought up and analyzed by engaging the teachers in solving problems.

In the following part we will describe two related issues that were found to be problematic in the discussions and analysis of ratio and proportion with the teachers. We have chosen to highlight them because research on ratio and proportion difficulties usually concentrates on other aspects.

Unawareness of the Different Intensive Ratios

A group of size N consists of two groups of sizes N_1 and N_2 such the ratio between N_1 and N_2 is a:b (where a and b are natural numbers that do not have a common divisor). If $N > a+b$ then it is possible to divide N into smaller and equal groups, such that the subgroups' elements conserve the a:b ratio. One possible alternative is to divide N into equal subgroups of size a+b.

For example: In a group of forty children consisting of $N_1=30$ girls and $N_2=10$ boys, the ratio between the total number of girls and the total number of boys is 3:1. The children can be divided into ten subgroups of four children, three girls and one boy in each group. The ratio between the number of girls and the number of boys in each of these subgroups is 3:1. It should be noted that the children could also be divided into five subgroups of eight children, six girls and two boys in each group. The ratio in each subgroup of eight is also 3:1.

The ratio a:b is an intensive ratio in Schwartz's terms (1988). This (same) intensive ratio exists between the original subgroups and also in the new subgroups. In order to describe teachers' conceptions we will use two different terms to talk about the (same) intensive ratio in different groups.

We will refer to the ratio between the sizes of the two groups, N_1 and N_2, as the 'global intensive ratio', and to the ratio between a and b within the subgroups of size a+b as the 'subgroup intensive ratio'. In the example, the ratio 3:1 is the 'global intensive ratio' between 30 and 10 and also the 'subgroup intensive ratio' in the small groups, between 3 and 1 in one division or between 6 and 2 in the second division. We show in this work that our teachers were not aware of the subgroup intensive ratio. A differentiation between intensive quantities is made by Kaput and West (1994), who distinguish between particular and rate conceptions of ratio intensive quantities. We have chosen to emphasize the different groups in which the intensive ratio appears, yet view the meanings of the ratios to be similar.

As mentioned earlier, the problem of dividing an amount by a given ratio is a common task for children in this subject. Keret (1997) lists several ways to solve it, including a trial and error method and some more systematic division methods. We have posed such a problem to our teachers, looked at their solution methods, and then analyzed different methods with them. The problem presented to the teachers:

A prize consisting of 840 IS is equally divided between children in a group. The ratio between the number of boys and the number of girls is 3:4. How much money will the boys get? How much money will the girls get?

All the teachers, except one, divided the total amount into 7 parts, calculated the total amount of 3 parts (the boys' share) and 4 parts (the girls' share). Their explanations were quite similar to each other. For example:

Simon: *840 is my whole. The boys have 3 parts of the whole, and the girls have 4 parts.*

Michal: *Each part is 1/7. The boys get 3x[(1/7)x840], the girls get 4x[(1/7)x840).*

Alice: *One part is 120, the boys get 4x120, the girls 3x120* [should have been the other way round].

Only Anna used a different strategy: *Each time [I distribute] I have 7. How many times do I have 7 in 840? [Then I calculate] how much altogether will the boys get? How much will the girls get?*

The first strategy, used by most of the teachers, can be regarded as a Partitioning Strategy. The number 7 stands for **the total number of shares** (parts), out of which the boys get 3 and the girls get 4. The second strategy can be regarded as a Quotitioning Strategy. The number 7 stands for **the amount of money given in each round** of the distribution, 3 IS to the boys and 4 IS to the girls. The use of this strategy involves the conception of the intensive ratio in the subgroups. All except one of the teachers were not aware of the subgroup intensive ratio in this problem.

Following a discussion on possible strategies, the teachers became aware of using their strategy in a technical way. They claimed that although they had used a Partitioning Strategy, they find the Quotitioning Strategy much more meaningful. It also occurred to the teachers that during the regular course of instruction they are working with children mainly on a quatitioning strategy. However, when given ratio-division problems, children are expected to use a technical partitioning strategy, which is not related to the way they developed their understanding of such tasks.

A more straightforward opportunity to exhibit awareness of the subgroup intensive ratio was given to the teachers in a discussion of the following problem:

A group of children consists of 32 girls and 52 boys. a) What is the ratio between the number of girls and the number of boys? b) Can the group be divided into smaller equal groups in a way that the ratio will be the same in all these groups?

The teachers had no trouble finding the ratio, and all teachers arrived at the answer 8:13 (although there are other correct answers). However, they all answered "no" on the second part. A reluctance to perceive the subgroup intensive ratio reappeared in a similar problem a week later in spite of an extensive analysis of the above problem and some additional examples presented to the teachers.

Over-generalization in Avoiding Additive Conceptions

According to what is known on the development of ratio and proportion concepts (Noelting, 1980a, 1980b) a common conception prior to the expert stage is the additive relation. Children, using an additive strategy, compare ratios by looking at the differences between the numbers within each ratio or between the two ratios. These children would judge two mixtures in Noelting's test to have the same orange flavor if they note an additive relation. They might notice the differences between mixture ratios and say that 2:7 is like 12:17 because 12=2+10 and 17=7+10, or they might notice the differences within each mixture and say that 2:7 is like 12:17 because 7-2=5 and 17-12=5.

Hart (1981) identifies the additive conception in a large number of students taking the ratio test in the Concepts in Secondary Mathematics and Science research.

An interesting difficulty, related to the additive conception, has emerged during our work with the teachers. We call it: 'Over-generalization in avoiding additive conceptions'. As it turns out, the (correct) knowledge that one should not compare ratios by using an additive strategy made our teachers so determined to avoid addition to the point that they claimed it is wrong to add even when it was a legitimate action. The following example demonstrates the teachers' difficulty:

This problem was presented in the workshop:

I went on a shopping spree in Kolbo making purchases in two department. The cost of the different items that I bought amounted to 900 IS. The ratio between my spending in these two departments was 2:7. I would like to pay my bill in three payments that do not have to be equal. Each payment consists of two sums of money for the two departments that keep the 2:7 ratio, and are paid in IS (in whole numbers).

List some payment plans that I can use.

The teachers suggested a variety of methods. Following a discussion on their methods, the teachers were asked: *Take one of the plans that you suggested. Now if you move 2 IS and & 7 IS from the two sums in one payment (choosing one that is big enough) and add it respectively to the two sums in another payment, will you get a plan that satisfies the problem?*

The group unanimously voted 'no' as an answer. Some examples of their reactions:

Ronit: *It does not keep the same ratio.*

Smadar: *But this is addition (!) It doesn't keep the same ratio.*

Michal: *I am sure it's not the same ratio.*

Alice (checking in her calculator): *I took 2 off the first amount to department A. That's 48. From 175* [payment to department B] *I took 7 off and got 162* [an error, it should be 168]. *I checked if the ratio is conserved and stays 2:7 and I found that it does not.*

In spite of this opposition and against her own intuition, one of the teachers makes the following statement:

Dorit: *After checking it with my calculator, [I have found that] it conserves the ratio, [this means that] there are many solutions. Because we are* *breaking up the payments that are built up of many 2 and 7-s. I know it is very difficult to get this. We [usually] think of ratio in the context of division and multiplication and not in addition and subtraction, that's why it is so difficult to think this way.*

DISCUSSION

The examples presented in this paper demonstrate teachers' inflexibility in understanding the ratio concept. This characteristic was exhibited on several occasions when, following a correct performance of conventional ratio tasks, the teachers were asked to look at things differently or try to modify their solution in order to get another answer.

In a ratio division problem most of the teachers solved the problem correctly using a technical partitioning method. Their method and the discussion they held following the task indicated a global conception of the ratio, i.e. a conception of the ratio as representing the relation between the two final parts into which the whole is divided. The teachers found it difficult to see the ratio as representing an intensive relation in smaller groups, and did not see the whole divided into smaller equal parts that conserve the same ratio.

This phenomenon repeated itself in several cases where the issue of finding smaller groups was the focus of an investigation. Although the investigation eventually led to more flexible conceptions, it took several examples to convince the teachers to 'see' the intensive ratio in different places.

Another demonstration of inflexibility appeared in the teachers' inability to move parts between subgroups. Their difficulty here might be a combination of their problem in perceiving the intensive ratio together with their reluctance to use an additive strategy. In their avoidance of addition they seem to be over-generalizing their knowledge that addition is incorrectly used when one is adding the same amount to the two parts of a ratio (or to the numerator and denominator of a fraction) in trying to get an equivalent ratio (or fraction).

In our effort to teach children the concepts of ratio and proportion we would like them to master these concepts with understanding. This research shows that first we might need to take care of some aspects of teachers' knowledge and help them increase their own flexibility in perceiving these concepts.

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PRESERVICE TEACHERS' CONCEPTIONS OF PROBABILITY IN RELATION TO ITS HISTORY

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Mathematicians, statisticians, and philosophers have had different conceptions of probability throughout its history, namely classical, frequentist, and subjective. Similar to this history, the findings of this study indicated that a group of 16 preservice teachers also held these three different conceptions of probability. However, whether their conceptions were classical, frequentist, or subjective mostly depended on the kinds of problems posed and their personal experiences. Moreover, their conceptions were often a mixture of classical, frequentist, and subjective. It is suggested that the teaching of probability should provide an opportunity for students to reflect on these diverse conceptions.

Diverse Conceptions of Probability: A Historical Account

Philosophers and mathematicians have held different conceptions of probability throughout history (Daston, 1988). At present, there are three distinct conceptions of probability that are derived from its historical tradition: classical, frequentist, and subjective (Borovcnik, Bentz, & Kapadia, 1991; Hawkins & Kapadia, 1984; McNeill & Freiberger, 1994; Steinbring, 1991).

Classical probability was first defined by Pierri Laplace (1749–1827) as "the ratio of favourable cases to the total number of equally possible cases" (Hacking, 1975, p. 122). Classical probability is obtained by making an assumption of equally likely cases and is sometimes referred to as *a priori*, Laplacian, or theoretical probability. Although, the credit for defining classical probability is given to Laplace, his definition was criticized because of the use of the word "possible," which has a dubious meaning. Contemporary mathematicians have refined this definition. For example, Kline (1967) expresses this as "*if, of n equally likely outcomes, m are favorable to the happening of a certain event, the probability of the event happening is m/n....*" (p. 524, emphasis in original).

Frequentist conception of probability was "connected with the tendency, displayed by some chance devices, to produce stable relative frequencies" (Hacking, 1975, p. 1). Some people refer to this as *a posteriori*, experimental, or empirical probability. It is defined as the ratio of the observed frequency to the total number of trials in a random experiment over the long run. Mathematically, it involves the theory of limits and convergence.

Subjective conception refers to probability as "evaluations of situations which are inherent in the subject's mind" (Borovcnik et al., 1991, p. 41). Daston (1988) defines it "as the intensity of beliefs" (p. 188). It assumes that human beings are capable of estimating the probability of certain events and making adjustments when additional data are obtained. It is also known as personal probability. Mathematicians and statisticians call it Bayesian probability, named after Thomas Bayes (1701–1761). Mathematically, it is associated with Bayes' theorem.

The different conceptions of probability that emerged have had significant influence on the teaching and learning of probability (Borovcnik et al., 1991; Hawkins & Kapadia, 1984; McNeill & Freiberger, 1994; Steinbring, 1991). Mathematicians, logicians, and philosophers have debated which of the views of probability is superior. For example, philosophers or mathematicians who believed that knowledge can be derived deductively in a formal fashion, argued for the classical view of probability. Those who believed in inductive reasoning supported the frequentist approach and those who believed in personal judgment supported the subjective approach to probability.

Preservice Teachers' Conceptions

Similar to the history discussed above, mathematicians, statisticians, and educators today have different views about the teaching and learning of probability in schools and universities. These differing views, especially the views held by preservice and inservice teachers, will have an important role on how teachers deliver their teaching in schools (Thompson, 1992). Studies, that focus on the different conceptions of probability (classical, frequentist, and subjective) are scarce. The purpose of this paper is to report about a study which investigated whether or not preservice teachers of secondary school mathematics held different views of probability, namely classical, frequentist, and subjective.

Methodology of the study

A total of 16 preservice teachers enrolled in secondary mathematics methods and problem solving courses at a Canadian University participated in this study. The participants were selected based on their mathematics and probability backgrounds as well as their interest in the study. Each preservice teacher had taken at least 10 mathematics courses and one probability course in his or her undergraduate degree.

Several items used by previous researchers to examine students' understanding of probability (for example, Konold, 1991) were reviewed. The items thus reviewed were modified for the purpose of the study. In addition, some new problems were also constructed using the investigator's personal experience of teaching probability. The study used the following two problems because of their potential to explore participants' diverse conceptions of probability.

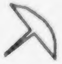
1. Lottery and car accident Problem

   Which one is more likely?

   (a) You will win a jackpot in Lottery 6/49.
   (b) You will be killed in a car accident.

2. Thumbtack problem

Each preservice teacher was shown a thumbtack and shown two positions of its landing on a wooden table, with the point down or with the point up as shown in the diagram below.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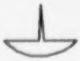
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When the preservice teachers observed the thumbtack the following question was asked: Which one (landing with the point down or point up) is more likely? Why? (Adapted from Konold, 1991)

In the first problem, the participants could use a combination formula to calculate the probability of winning the jackpot in the lottery 6/49. However, no formula could be used to calculate the probability of being killed in a car accident. They could not use such a formula in the second problem as well.

Both problems were posed in an interview setting. The participants were probed to determine their conceptions of probability (classical, frequentist, and subjective). For example, if the participants made certain assumptions and used the classical view of probability they were asked whether or not they believed in conducting a random experiment to determine probability and the vice versa.

The participants' responses to all these problems were audiotaped, transcribed, and then analyzed in terms of classical, frequentist, and subjective probability. If the participants made certain assumptions to determine the probability of an event, it was categorized as classical. For example, in the lottery and car accident problem, if they assumed that the probability of any six numbers being selected in the lottery is equal, and calculated the probability based on the combination formula $C(n,r) = \frac{n!}{(n-r)! \times r!} = \frac{49!}{(49-6)! \times 6!}$, the response was called classical. Conception was called frequentist if the participants wanted to know how many people in the past have won the jackpot in 6/49 lottery in order to determine probability by taking a ratio of the winners to the total number of people who played the lottery. Responses were called subjective if the participants stated that the probability of being killed in a car accident is higher than winning the jackpot in the lottery because the number of people killed in car accidents is higher than the number of people winning the lottery.

Results

Although philosophers, logicians, and mathematicians favored one conception or the other during the evolution of probability, the preservice teachers seemed to hold different conceptions of probability simultaneously. Nevertheless, the preservice teachers' beliefs to classical, frequentist, and subjective conceptions differed based on the kinds of problems on the one hand and their personal experiences and intuitions on the other.

In the lottery and car-accident problem, 14 out of 16 participants used a classical conception when they tried to determine the probability of winning the lottery 6/49 by using the combinatoric formula. They used the formula and calculated that the probability of winning the jackpot in the lottery was about 1 in 14 million. In the same problem, they used a mixture of classical, frequentist, or subjective conception to determine the probability of getting killed in a car accident. Nine out of 16 preservice teachers wanted to use data provided by the investigator related to death in car accidents in the Canadian province of British Columbia. They calculated the probability by dividing the number of total deaths in car accidents by the total number of people living in the province. Based on these calculations, they concluded that the probability of being killed in a car accident was higher than winning the jackpot in the lottery 6/49. This response can be classified as a mixture of classical and frequentist because they were calculating a ratio based on data available to them making an assumption that every resident of the province had an equal chance to become a fatality.

The remaining seven said that they did not need data for this problem because the probability of getting killed in a car accident was much higher than winning a jackpot in the lottery 6/49. When asked to provide their reasoning they stated that they had read or watched news about killings in car accidents almost everyday but there was hardly any news about winning a jackpot in the 6/49 lottery. Hence, these seven preservice teachers were using their subjective conceptions of probability.

Different conceptions of probability held by preservice teachers were also observed in the thumbtack problem. Four preservice teachers wanted to run the experiment with the tack before making any decision. They threw the thumbtack and recorded the frequency of landing "point up" and "point down". They believed in a frequentist conception and obtained the probability of point up or point down by dividing their total frequencies by the total throws. Interestingly, three of these four preservice teachers had used the combination formula to calculate the probability of winning the jackpot in the previous problem. Hence their choice of conception was based on the kinds of problems posed. When probed, they stated that they did not know any formula to apply in the thumbtack problem.

Twelve preservice teachers used the structure of the thumbtack to decide whether the tack would land "point up" or "point down." They basically began with a subjective conception and stated that "point up is more likely because the flat side of the tack is heavier and more stable than the pointed part." Nine out of these 12 participants wanted to use a classical conception of probability, but were unable to determine the odds of point up to point down based on some assumptions. They then stated that they would run the experiment with the tack after some thoughts about its landing. They initially based their view on a classical conception.

However, they stated that they would change their initial view of likelihood based on subsequent trials and experiments. For these preservice teachers, classical and frequentist conceptions of probability go hand in hand helping to modify their thinking processes. That means they viewed both deductive and inductive thinking as being helpful in making decisions regarding probability.

About half of these 16 participants, however, were not sure about using a frequentist conception of probability because they could not decide how large a sample is large enough. For example in determining whether the tack would land point up or point down, the concept of large numbers varied from person to person. Four participants stated they should try dropping the thumbtack about 100 times to get a rough estimation of the landing of the thumbtack. Three other argued that they should drop the thumbtack 1000 times. For two other participants, even 1000 trials would not have been enough. These two participants stated that they would not trust even a thousand trials. According to them, they would conduct a computer simulation and run this experiment a million times. Basically these preservice teachers did not prefer a frequentist approach. A computer simulation is developed based on a classical conception because of certain assumptions used in the model.

Despite the preservice teachers' various beliefs about how much would be considered a large number of trials, all the preservice teachers believed that the larger the number of trials, the better the estimates. For example in the thumbtack problem, one participant stated the logic as follows:

> The fewer trials you do the more error you're going to have in your assumption.... You don't want to sit there for an hour and drop it. But we do it 10 times [drops the tack ten times] See, that says 90%. But you know if you do it another 10 [drops the tack ten times again]. That was 70% for this one. So I take the average of the two. Based on 20 trials these are the odds. But the more trials you do the closer you're going to get the actual probabilities.

Another participant stated that he would start with 1000 trials to come to an estimate. He would then increase the number of trials and adjust his estimate. But in general, the number of trials depended on how they viewed the likelihood of the problem before actually trying the experiment. For example, one participant stated that she would do the experiment to verify her theory that the thumbtack is more likely to land point up. But she would not change her belief if the thumbtack lands with point down 60 times in a throw of 100 times. She would not discard her theory even with 1000 trials. But, some other participants would constantly change their estimation based on the experiment. Whether or not the preservice teachers would constantly change their beliefs based on an experiment depended very much on whether they believed in a classical, frequentist, or a subjective view of probability. The classicists did not believe in changing their theory unless the experiment is conducted a large number of trials like 10,000. The frequentists and subjectivists constantly changed their theory based on experimental results, even with 10 trials.

Conclusion

Although the preservice teachers had their preferences as to whether they would trust more a classical, frequentist, or subjective conception of probability, they did not blindly follow one particular conception or the other. As indicated above, their conceptions largely depended on the nature of the problem and their prior experiences related to the problem. For example, the participants demonstrated a classical conception in the lottery problem because they were familiar with the combination formula. However, many of the same participants demonstrated a frequentist or subjective conception in the car accident problem even though the lottery and car accident problem was provided as one task. By the same token, the participants used mostly a frequentist or a subjective approach to the Thumbtack problem because no immediate mathematical formula was available to them.

While the preservice teachers' conceptions were mostly based on kinds of problems asked, they had a preference whether they believed more in a classical, frequentist, or subjective conception of probability. The majority of the participants always attempted to solve the problems using a classical approach. They switched to a frequentist or a subjective approach only when they could not use a probability formula making some assumptions. The other participants preferred to conduct an experiment and update their thinking to determine probability. Their conceptions could be summarized with Figure 1.

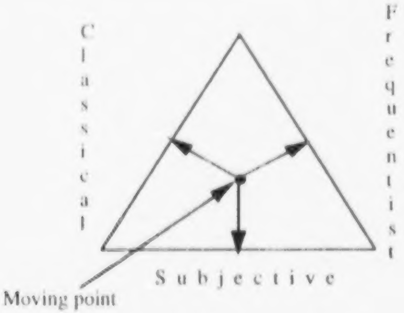

Figure 1. Preservice teachers' differing conceptions of probability

The three sides of the triangle are represented by classical, frequentist, and subjective conceptions of probability. The point where the three arrowheaded lines intersect represents a preservice teacher's conception. The point is not always at the same place for each preservice teacher. Rather, the point moves from person to person and within a person. For some of the preservice teachers, the point moves towards a classical conception of probability. But that does not mean that they do not hold any frequentist or subjective conceptions of probability. Similarly a preservice teacher may have more faith in a frequentist conception, but he or she may still hold a classical or subjective conception of probability.

Once again the preservice teachers' conceptions of probability were not purely classical, frequentist, or subjective at all times, but their conceptions were classical on one problem, frequentist on another, and subjective on other problems. Further, some of the preservice teachers' conceptions were a mixture of classical, frequentist, and subjective and at times it was difficult to separate one from the other.

Teaching Implications

The debate about different kinds of probabilities: classical, frequentist, and subjective has influenced the research and teaching of probability. Presently, mathematics and statistics educators have contrasting views as to whether a classical, frequentist, or subjective view of probability should be taught (Borovcnik et al., 1991; Hawkins & Kapadia, 1984; Konold, 1989, 1991; Shaughnessy, 1992, Steinbring, 1991). This is basically an epistemological debate, but teachers face difficulties in determining appropriate pedagogical approaches for teaching probability when these epistemological issues are unresolved.

The results of this study indicate that preservice teachers hold various conceptions of probability simultaneously. These differing conceptions held by preservice teachers are not necessarily incorrect. Rather, the conceptions are valuable based on problems posed. A classical approach can be more valuable in one kind of problem whereas a frequentist or subjective conception can be valuable for other kinds of problems. The classical approach to teaching probability is powerful in solving problems in which assumptions can be made more easily. The subjective and frequentist approaches to teaching probability are particularly useful when assumptions about a problem cannot be determined easily. Moreover, these approaches provide students with the opportunity to update the probability of a certain event according to new information.

In some cases, all different conceptions of probability can be used to solve the same problem. For example, students can solve the lottery and car accident problem using all three approaches and determine their effectiveness. Students are more likely to develop a conceptual understanding of mathematics if they are provided with the opportunity to use varieties of approaches. The teaching of probability only through one approach is harmful for students' conceptual understanding and so a multiplicity of approaches depending upon the situations at hand should be utilized.

Another implication of this study is that teachers of probability should provide students, with varieties of problems that can be solved by using different approaches. Students' differing conceptions that emerge in class should be discussed in terms of their strengths and weakness in solving future probability problems. Such discussion can prepare students in solving varieties of problems in future.

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Team Teaching and Preservice Teachers' Classroom Practice with Innovative Methods of Instruction –The Case of Computers

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Abstract

This paper reports on the effects of team-teaching on preservice teachers' classroom practice when implementing an innovative computer-based program. Novice student teachers, working in pairs, taught a mathematics class using computers. They were generally afraid of computers and had not expected to teach with them. However, after this teaching experience, all the preservice teachers had high expectations of computer use in their classes. They reported that team-teaching reduced anxiety, gave them confidence and helped solve discipline problems.

Research has indicated that student teachers generally adhere to the classroom practices and strategies they themselves experienced as school pupils (Kagan, 1992). Thus, for innovative methods of instruction to be adopted and implemented by these future teachers, their perceptions of teaching and learning have to undergo some change. Computers are clearly an important medium for teaching mathematics. Working with computers requires a change in the teacher's role in a computer–classroom; however this causes anxiety for practicing teachers (Ainley, 1996). Despite increased availability and support for computers, levels of computer use among teachers are not very high. With regard to preservice teachers, their expectations of computer use are high, however these expectations drop after their first year of teaching (Marcinkiewicz, 1996). Effecting changes in teaching practice is difficult (Levenberg & Sfard, 1996) but possible (Bednarz, Gattuso, & Mary, 1996), where a necessary condition for change in teaching patterns is reflective thinking on the part of the teacher (Schön, 1983). Teachers play an important role in introducing innovative programs since they are the filters between the program developers and the students. (e.g. Moreira, & Noss, 1995). Team teaching facilitates professional development and effecting school change (Griffin, 1991). Other benefits that have been cited through team teaching are a reduction in teacher isolation (Chazan, Ben-Chaim & Gormas, 1996), improved instruction, a decrease of disciplinary problems in the classroom and greater teacher motivation (Spies, 1995).

This paper reports on a case study of a class of 15 novice preservice teachers who taught in pairs (one novice teacher was concurrently a member of two teams) and implemented an innovative mathematics program that relied heavily on the use of computers in junior high school.

Background

The school requested that the preservice teachers teach the topic of statistics to seventh grade students. Most of the students did not feel at home with a computer. Some students barely knew how to use the computer for word processing - the most commonly used feature of computers in the college - let alone the electronic spreadsheet. So when the students were informed that they were to use an innovative program which taught statistics through the medium of the electronic spreadsheet, the students were understandably alarmed. They were required to teach mathematics for the first time in a classroom situation, with all the accompanying anxieties, and with a computer to boot – a tall order! In this college, a 'didactic' instructor accompanies a group of students in their practice teaching. The instructor meets with his or her group in a joint weekly session where discussions are held on the current topic that the student teachers are teaching in their classrooms. The instructor examines the students' lesson plans and then regularly observes the students in their classroom practice for at least part of their lessons. Notwithstanding this quite intensive attention and follow up, the first year of practice teaching always holds problems for the student teachers – discipline problems, instruction problems and the like. As mentioned above, student teachers tend to perpetuate the practices and strategies that they experienced as school students and do not voluntarily integrate technology into their classroom practice (Marcinkiewicz, 1996). Since teachers' colleges obviously want to train their future teachers to be as up to date as possible, the instructor felt that it was important for student teachers to learn and experience new classroom teaching strategies in a computer environment while they were in preservice situations. The novice teachers might then enter the teachers' work force with experience and confidence in more innovative teaching strategies and have positive expectations for using technology in their classrooms. However, too many new challenges and changes might be counterproductive and thus jeopardize the novice preservice teachers' chance for succeeding. The conjecture was that team teaching would offer the novice student teachers support in this innovative computer-teaching experience and allow them a measure of success and confidence that they might then take with them into their future classroom practice.

Study Design

It was decided that these student teachers would team teach in pairs. Both teachers would always be present in the classroom but would take turns in being the "active" teacher. While one teacher would expound on a topic to the whole class, the other would listen, observe his colleague's didactical capabilities and note any difficulties that might later need be attended. However, during the time when students would work on assignments both teachers would have the same role in the classroom: assisting the students, resolving any problems encountered and the like.

The data collected in this study were transcripts of semi-structured interviews carried out with all preservice teachers towards the end of the school year. There was also a follow up interview with one of the preservice teachers at the beginning of his second year of preservice teaching. This preservice teacher, Uzi, figures prominently in this paper as his case was particularly interesting since he had originally expressed considerable anxiety at the idea of computer-medium teaching. The data was analyzed in order to assess the effect of team teaching on various aspects of the preservice teachers' instructional experience in their first year.

Results

Reduction in teacher isolation: The first benefit from team teaching noted by many of the student teachers was the confidence they gained in being able to share their new, daunting experience with another teacher.

Uzi: When I first heard that I was going to work with computers my hands and feet shook....But when I was told that I was going to share my work with another person it calmed me, it really calmed me down. And why did it calm me, because if I don't know (something) *there is some one who can help me now... There is a certain collaboration and not everything falls on me.*

Lior: Team-teaching – first and foremost this gives confidence.

Orna: ..When I heard computers I was a bit frightened. I didn't come with a background in computers. And when I heard that there is teamwork, I calmed down.

Discipline: Apparently novice preservice teachers may find difficulty in focusing concurrently on both teaching the subject matter and taking care of discipline problems. All the preservice teachers in this study found the necessary support in their team teacher. For example:

Dina: It's difficult to keep the quiet (in the class) *and also to teach the material.... There is less pressure on one teacher, both to take care of the discipline and to teach the material.*

Orna: When there is someone, someone else with you, they (the other team teacher) *know exactly when to approach the specific student who's causing a disturbance and to quiet him down.... or when one has actually to stop the lesson because none of the class is really with us.*

Later in the year, Boaz and Basil were having some discipline problems so they decided that each teacher would teach half their class on his own. Finding that working apart was less efficient, they rejoined. Boaz summarized the differences:

Boaz: When I worked with him (Basil) *we would be in contact with each other all the time. Basil, what should we do here, how shall we start, how shall we end? There was always someone who could give you* *answers...And when we were working on our own it was like, that's it, now you're on your own. Start coping by yourself... And it* (the usual communication) *was lacking, very lacking. But on the other hand it gave me the advantage of entering a class alone and beginning to cope.*

This was a theme that came up constantly: the preservice teachers were aware that future teaching would most likely take place in a single teaching situation and they felt that team teaching was a non-realistic utopia. However, as illustrated above, they clearly felt relieved when they learned that they would not be so isolated, that they were to share their new experience with another teacher - someone who might offer them support and solutions in times of need. The team-teachers spoke of the "fun" it was, working together – motivating them to invest time and effort in lesson planning. Apparently this sense of sharing was so positive that the teachers of one of the teams, who each were also meant to teach their own physics class in this school, of their own volition adopted team-teaching in their physics classes.

Researcher: Would you have liked, let's say, to have started to teach also your physics class as a pair?

Uzi: Yes, and it happened! We (team-) *taught without the didactic instructor knowing... When she wasn't here* (at school) *we used to go in as a couple. Right at the beginning* (of the school year) *and no one knew about it... Lior and I would go in and teach the lesson together, until it came to be that if it was Lior's lesson then Lior would teach and I would observe, until we, like, got accustomed. The tension eased, we felt a little more confident, and that was it. Each one spread his wings...*

This physics team-teaching revelation was indeed a surprise. One may wonder why they had not discussed team-teaching with their physics didactic instructor. Since this team-teaching occurred at the beginning of the year, one can only surmise they still felt that team-teaching was a somewhat irregular teaching method. On the other hand it seems they felt team-teaching to be so effective and supportive that they did not want to jeopardize their chance for using it by receiving a possible negative response from their instructor.

Expected teaching methods in classroom practice: Almost all the preservice teachers had expected to apply the same teaching methods to their classes as those their teachers had used with them when they were school pupils. When asked whether they had expected to use computers in their mathematics class, 14 preservice teachers responded with an unqualified "No!" :

Uzi: First of all I thought that I would teach as I was taught...chalk, me and the blackboard and the students, work sheets.... I wished on myself that I would never need the computer for teaching... I always thought of *how I was taught and it* (the teaching method) *was easy... because this method was indeed successful and I see myself as one who matriculated... without a computer.*

Yifat: When you came and told us that we were going to teach statistics via computers I was most surprised. Because I was never taught like that...I like the computer very much (but) *at the beginning I was apprehensive of teaching with computers.*

Shirley: I didn't learn (in school) *with a computer and I saw that it worked out and I didn't feel that anything was lacking. And I have a computer at home that has been sitting there for years, and I never used it. Maybe we played games on it, nothing more. And when I didn't think of teaching with a computer... At first, when you spoke about* (teaching with) *a computer I thought, oh no – computer – I really dislike computers. I have no patience* (for it). *But today... for example I see it as something positive, because first of all it gives the children some variety in the lesson. Simply, until I would have experienced it myself* (teaching with a computer) *and realized I wouldn't have tried* (teaching with computers).

The above quotes support research indicating that student teachers would adhere to teaching strategies that they themselves experienced as pupils. It would seem, then, that in this vein these student teachers would not expect to implement innovative teaching practices – in this case, use of computers in their classrooms. And indeed all but one of the 15 student teachers had unequivocally not expected to teach with computers, thus reducing support for claims that preservice teachers' expectations of computer use are high.

Synergy and Reflective-team-teaching: In order for the novice teachers to implement computer use in their classes, it would seem that certain models of what they conceived as teaching might have to be modified. Could reflective thinking, in particular "team-reflective-thinking" as opposed to self-reflective-thinking, facilitate this process of change, and if so how? The following is an excerpt of what Lior thought, when expounding on the contribution of team-teaching:

Lior: Every student has his own ideas, and there are always differences from student to student. The minute there is a meeting of two (people) *then one makes a meeting between two different ideas that maybe the students did not think about... And then I adopt it for myself, ideas, perhaps work methods that one can take from another student. It* (team-teaching) *gave me a tremendous amount...* (For instance) *every student* (teacher) *has a case when something happens to him* (in class), *and you know how you commented "Lior, when you get upset, when you're hot-tempered...", and I got this sort of look* (from my team-teaching partner) *and no one in the class recognizes this look. And then Lior calmed down, kept quiet, and it* (this event) *stopped... So the partner serves as a sort of online criticism... And there's another thing. I believe that when you have a partner, reflective thinking happens willy-nilly. That is, you go back* (together) *by bus, by car, you're compelled to speak about what has happened. Not because you're required to think reflectively, but you speak about how it was, what was, what we should have done. That we understood them, they enjoyed, did not enjoy, what we should do next time... If I was teaching on my own I would have to sit and think about what had happened in the lesson. The minute we* (the team-couple) *speak about it,... we remember it well."*

Lior depicted an ongoing situation, prevalent in all the teams, wherein team teaching offered exchanges of ideas, continuous feedback both in class and out, resultant modification of strategies and consequent improved instruction. His description of their rich, very effective reflective-team-thinking as finding its immediate expression in speech seems to echo Vygotsky (1986, p 251)) who wrote "...thought does not express itself in words, but rather realizes itself in them." This realization of reflective-thinking seemed to afford the teachers a very effective instrument for change. The novice preservice teachers were faced with continual teaching challenges that might evoke in them inappropriate reactions. Their team-partners were there as safety nets, perhaps rescuing them from some situations that might have produced feelings of incompetence or inadequacy in them. The immediate reflective-team-thinking that followed the lessons allowed the teachers to analyze their lessons, to discuss their actions and reactions and when necessary to design different strategies for similar, recurring circumstances. This would seem to have been especially crucial when working with computers. The student teachers had to create new teaching patterns since they had no teaching models that they had experienced as school pupils on which they could fall back. The synergy in the teamwork seemed to offer the student teachers an opportunity for creating successful teaching patterns.

Computer use in future classroom practice: Prior to this teaching experience the novice student teachers had generally not expected to use a computer in their classroom practice. However, at the end of school year their expectations were quite different:

Orna: The children themselves were very enthusiastic (when learning statistics with the computer)... *I would be very interested in integrating it* (the computer) *together with the* ("conventional") *learning, like we did now. There were also lessons when we sat in class and learned, and parallelly there were also lessons when we learned with the computer. I think that this combination between the two was excellent.*

This is the same Orna who had expressed anxiety at the idea of teaching with computers. It seems that successful experiences might assist teachers to form new normative patterns of teaching. Uzi allows a very close look at a teacher, who came in with much trepidation to teaching with computers and who now had no qualms about future computer use. The following school year Uzi was student teaching without computers. He approached this researcher saying that he missed last years' teaching very much; consequently a follow up interview was performed:

Researcher: Which teaching (do you miss)?

Uzi: That I taught with a computer... I know that computers are taking an important part in the teaching process... and I said to myself this (conventional teaching) *is going to change at some stage... We're required to do our seminar papers* (in the teachers college) *with computers, and all the statistical analyses...*

Researcher: If you yourself wouldn't have experienced (computer teaching), *I'm asking you once again, do you think that despite the importance that you are being shown,... even though you see its importance, would you have postponed* (teaching with computers)?

Uzi: I would have postponed... because I'm afraid of computers. I was afraid. It is something deterring... Nevertheless, last year I went in (to computer teaching), *but there were many things that I didn't know how to operate, even though the pupils didn't know, it bothered me... It is like, for instance in mathematics, to teach 5 units* (the highest level) *mathematics. Never in my life would I be afraid of standing before a twelfth grade class and teach them, because I know everything. There is nothing they know that I don't know because I went through it, I experienced it...* (Nevertheless) *I want and am willing to go to teach with computers,... because of the experience.*

Evidently Uzi still did not feel entirely comfortable with teaching with computers. This, despite him having been exposed a considerable amount to the computer at the college. Uzi felt inadequate teaching with computers since he did not master the computer from every possible angle. However, it seems that having had a successful teaching experience, perhaps having also seen how pupils responded positively to learning with computers, was enough to swing the balance from being a teacher quite vehemently opposed to teaching with computers to being quite positive about teaching with computers.

Conclusion

This study sought to explore the effects of team-teaching on preservice teachers. Team-teaching proved to be a supportive framework for the new teaching experiences that the student teachers encountered – developing their teaching strategies and reducing computer-teaching related anxiety and discipline difficulties. Reflective-team-teaching seemed instrumental in allowing the novice student teachers to modify familiar teaching patterns and thereby successfully implement an innovative, computer-medium statistics program. Following this team-teaching experience, the preservice teachers' expectations of computer use in class changed from low to high. They had become future teachers who looked forward to integrating computer technology into their classrooms. It may be thus concluded that team-teaching might be an effective instrument for successfully instituting innovative methods into preservice teachers' classroom that they will take with them in their future teaching. Further research is required to examine to what extent these teachers will indeed implement these innovative instructional methods in future practice.

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PROFESSIONAL INFLUENCES ON TEACHERS PERCEPTIONS OF TEACHING AND ASSESSMENT IN MATHEMATICS

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ABSTRACT

This paper reports and analyses findings from an investigation into the effect of INSET upon teachers' perceptions of teaching and assessment in Mathematics. Questionnaires were distributed to Cypriot teachers before and after the attendance of an INSET course on policy for curriculum reform in Mathematics. The main research findings are the following. After the attendance of the INSET course, teachers considered the purpose related to mathematical communication as particularly important. Moreover, after the attendance of the INSET course, teachers considered pupils' abilities to apply their knowledge in unfamiliar situations and pupils' attitudes to Mathematics as important objectives for teachers' assessment. They finally considered oral techniques of assessment as more appropriate than written techniques. However, their perceptions about classroom organisation and the development of school policy did not change. Implications for the development of policy on teachers training are discussed.

I) INTRODUCTION

Changes in the teaching of Mathematics have always been in the heart of curriculum reforms. Until recently most of the efforts in trying to accomplish desired changes centred on the development of improved curricula. However, the failure of much curriculum innovation in Mathematics can be attributed to the neglect by innovators of teachers' perceptions. A long trend in the literature (Fullan 1991) supports the view that teachers' perceptions are one of the most critical factors for the effectiveness of the various models of curriculum change. The importance of teachers' perceptions is also supported by research on teachers' thinking (Calderhead 1987, Clark 1988). Although such research does not provide us with a comprehensive and theoretical framework for thinking about teaching, it does provide us with an insight into the process of curriculum change. Calderhead (1987) points out that research into teachers' thinking shows:

> "how unrealistic it is to conceive of innovation as a set of pre-formulated ideas or principles to be implemented by teachers. Innovative ideas are interpreted and reinterpreted by teachers over a period of time and translated into practice in a process that involves teachers drawing upon several different knowledge bases and interpreting and manipulating various interests" (p. 17).

Teachers possess a body of specialised knowledge acquired through training and experience related to teaching methods, subject matter and child behaviour together with other information resulting from their experience of working with children in numerous contexts. Understanding the factors influencing teachers is necessary for any attempt to evaluate curriculum reform in Mathematics.

Investigation of mathematics teachers' perceptions has become a significant endeavour in recent years (Pajares 1992, Malone 1996). However, Grows & Schlutz (1996) point out the lack of research on mathematics teacher development. Little information is available relating to the effects of professional training upon teachers' perceptions. Nevertheless, studies of teacher-training programmes reveal that teachers are generally offered little opportunity to change the views they formed of mathematics and how it is taught and learnt during their years of pre-university schooling (Bednarz et al 1996). Grouws & Schultz (1996) argue also that there was little information available about the overall design features of in-service education programmes which produce changes in teachers beliefs and classroom practices. This raises the question of how to bring about the necessary changes in the way teachers view mathematics teaching which will be fundamental in their future practice. In this context, the purpose of my research was to investigate the extent to which the attendance of an INSET course could influence Cypriot teachers' perceptions of curriculum policy in Mathematics.

II) METHODS OF DATA COLLECTION AND SAMPLING

In Cyprus, in 1994, a reform programme in Mathematics was introduced which was mainly concerned content, pedagogy and assessment. A centre-periphery model of change was used. The central government, through inter-departmental committees, drew up syllabuses, curricula, and planning guides, which were distributed to schools. However, policy makers did not take into account the need for a strong link between curriculum reform and teacher development which is reflected in theories of curriculum change (Fullan and Hargreaves 1992). Thus, a teacher training programme was developed around the main policy initiatives on teaching and assessment in Mathematics. One of the objectives of this course was to expose teachers to an approach to Mathematics that is different from the one they have experienced previously, and to introduce them in a problem solving context to another teaching culture by instituting a process of explanation, discussion and negotiation within the classroom. The activities offered in this course did not only enable teachers to understand the policy initiatives on teaching and assessment in Mathematics but also provided them the opportunity to "reflect in and on action" (Schon 1983). The extent to which the attendance of the course helped them to change their views of teaching and assessment in Mathematics is therefore an issue which has to be explored.

Teachers who attended the course on teaching and assessment in Mathematics were asked to complete a questionnaire both before and after they had attended the course. The content of the questionnaire was derived from analysis of policy for curriculum reform in Mathematics in Cyprus. There were five broad areas of teachers' perceptions:

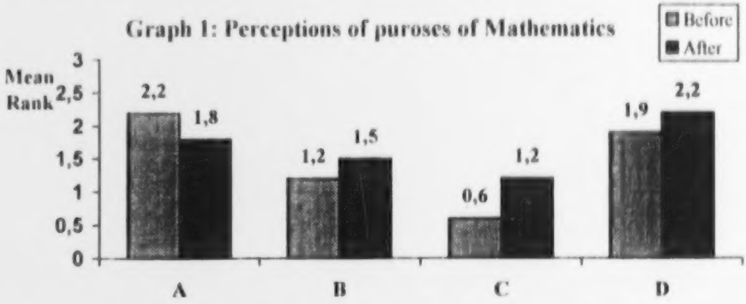
a) The purposes of teaching Mathematics,
b) The purposes which assessment should serve,
c) The relative importance attached to different teaching methods,
d) Techniques of assessment,
e) Ways of improving assessment

Of the 287 teachers approached 282 responded to the questionnaire given to them when they begun to attend the course and 279 responded to the questionnaire after they had attended the course. Although it was not possible to identify those teachers who answered the questionnaire twice since the questionnaires were answered anonymously, the fact that the response in both cases were particularly high (98% and 97% respectively) and the questionnaires were administered to the whole population, implies that figures derived from each sample can be used for generalisation to its population. Thus, this comparison may measure changes in perceptions of teachers who attended the course on Mathematics.

III) FINDINGS FROM THE QUESTIONNAIRE

Purposes of teaching Mathematics

Graph 1 shows the mean rank of the perceived importance of each of four purposes of teaching Mathematics. Kendall Coefficient of Concordance was calculated to show the degree of consensus about curriculum purposes in this ranking. A significant level of agreement amongst teachers before ($W_1=0.31$, $p<.001$), and after the attendance of the course ($W_2=0.35$, $p<.001$) was revealed.



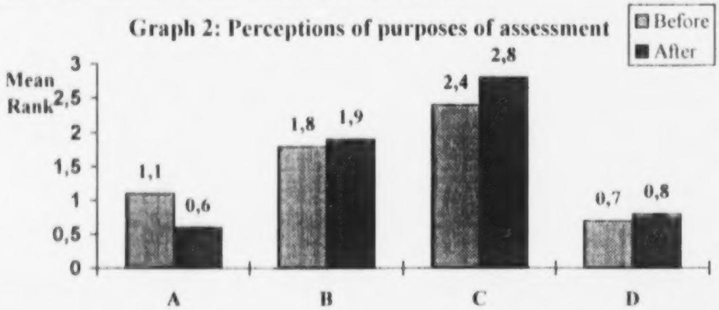
Graph 1: Perceptions of puroses of Mathematics

A: Gain knowledge, B: Positive attitudes, C: Talk about Maths D: Solve problems

The following observations arise from Graph 1. Before the attendance of the course, teachers gave high priority to purposes concerned with gaining Mathematical knowledge and solving investigative tasks. The purpose which was focused on pupils` ability to talk about Mathematics was seen as the least important. After the attendance of the course, they considered equally important all the purposes of Mathematics. Thus, the Wilcoxon test revealed that after the attendance of the course teachers considered less important the purpose concerned with gaining mathematical knowledge ($Z=2.8$ $p<.01$) and more important the purpose focused on pupils' ability to talk about Mathematics ($Z=3.2$ $p<.001$). This finding can be linked with the fact that one objective of the course was to enable teachers to see Mathematics as a language.

Purposes of Assessment

Graph 2 deals with perceptions of purposes of assessment. Kendall coefficients of concordance for teachers' perceptions about purposes of assessment revealed a significant level of agreement amongst teachers in their ranking of the relative importance of the purposes of assessment both before ($W_3=0.51$, $p<.001$), and after the attendance of the course ($W_4=0.64$, $p<.001$).


Graph 2: Perceptions of purposes of assessment

A: Suminative B: Teacher's self-assessment C: Formative D: Evaluative

The following observations arise from Graph 2. Before the attendance of the course, teachers considered formative assessment as the most important purpose and teachers' self-evaluation as the second most important purpose. It is also of interest to emphasise the low rating given to summative purposes of assessment and to the national monitoring. Thus, summative purposes and national monitoring were considered as the least important purposes of assessment. Although purposes of assessment were ranked similarly by teachers before and after the attendance of the course, the Wilcoxon test reveals that after the attendance of the course teachers considered formative assessment as more important purpose ($Z=2.7$ $p<.01$) and summative assessment as less important ($Z=2.8$ $p<.01$). These statistically significant differences can be attributed to the fact that after the attendance of the course almost all teachers (85%) considered formative assessment as the most important purpose and summative assessment as the least important purpose.

Methods of teaching and assessment in Mathematics

The figures in Table 1 are based on the information derived from teachers' responses to items of the questionnaire concerned with the implementation of policy on Mathematics pedagogy and assessment. It illustrates all the statistically significant differences between teachers' perceptions before and after the attendance of the course. The following observations arise from Table 1. After the attendance of the course, teachers supported as a group those methods of teaching and assessment in Mathematics which before the attendance of the course were not considered as appropriate. Thus, before the attendance of the course 48% of teachers did not agree that practical activities are appropriate for older and high attainer pupils. But after the attendance of the course, only 25% of teachers did not agree with these two methods. Moreover, before the attendance of the course, 28% of teachers did not support that pupils should talk about Mathematics and present the results of their activities to their classmates. However, after the attendance of the course, almost all of them (88%) agreed with this method. The last two items of Table 1 are concerned with issues of assessment policy in Mathematics. After the attendance of the course, the great majority of teachers (85%) considered pupils' ability to apply Mathematics in unfamiliar situation and pupils' attitudes to Mathematics as important objectives for teachers' assessment whereas before the attendance of the course more than 25% of teachers did not agree with these two items.

Table 1: Means, standard deviations and t-values derived from comparisons of teachers' perceptions before and after the attendance of the course

Methods of teaching and assessment in Mathematics	**Before**		**After**		**t**	**d.f.**	**ñ**
	Mean	**S.D.**	**Mean**	**S.D.**			
Practical activities as appropriate for younger as for older pupils	2.88*	0.93	3.45	0.76	2.72	275	.01
Practical activities as appropriate for high attaining pupils as for low	2.86	0.83	3.41	1.02	3.72	272	.001
Needs for talk in each activity	3.42	0.93	4.01	0.76	3.31	274	.01
Assessment of pupils' attitudes to Mathematics	3.18	0.93	3.95	0.86	2.67	268	.01
Assessment of child's ability to apply Mathematics in unfamiliar situations	3.02	1.13	3.88	0.96	2.66	271	.01

*1. Absolutely disagree, 2. Disagree, 3. I do not know, 4. Agree, and 5. Absolutely agree

It is also important to note that no statistically significant difference was identified between teachers' responses to items concerned with the need for developing a school policy on teaching Mathematics before and after the attendance of the course. A great variation among teachers views of the development of school policy in Mathematics was identified both before and after the attendance of the course. Moreover, most teachers, both before (89%) and after (87%) the attendance of the course, revealed that they organised their mathematics lessons in such a way that children spend most of their teaching time in working as a whole class. Cluster analysis was used to identify relatively homogeneous groups of teachers according to the proportion of time in Mathematics lessons that their pupils spend in working on individual tasks, on collaborative group tasks and as a whole class. No group of teachers organised their mathematical lessons in order to distribute their time equally between the three ways of classroom organisation.

Techniques of Assessment (Appropriateness and Ease)

Teachers were asked to rank twice eight techniques of assessment in Mathematics according to their appropriateness and their ease. Kendall coefficients of concordance revealed that both before and after the attendance of the course teachers agreed among themselves in their ranking of the relative appropriateness of each technique (Before: $W_5=0.18$, $p<.001$; After: $W_7=0.21$, $p<.001$) and also agreed among themselves in their ranking of the relative ease of each technique (Before: $W_6=0.27$, $p<.001$, After: $W_8=0.23$, $p<.001$). However, the mean ranks tend to cluster close to each other, with small differences between them. Nevertheless, the Wilcoxon test shows that after the attendance of the course, teachers considered multiple choice and matching question as less appropriate technique than before the attendance of the course ($Z=2.7$, $p<.01$). The eight techniques of assessment in Mathematics were also classified into two categories, namely oral and written techniques. The category of written techniques represents an average of the methods which have to do with a written test and the oral category represents the rest of the techniques. The Wilcoxon text revealed that after the attendance of the course, teachers considered written techniques as less appropriate techniques than before the attendance of the course ($Z=2.6$, $p<.02$). It is also important to note that both before and after the attendance of the course three of the oral methods were considered as the three most appropriate techniques whereas unstructured observation was considered as the least appropriate technique. This raises a question about whether the oral category is a coherent one on teachers' perceptions. It can be also argued that the three oral techniques which were considered as the most appropriate (Structured observation, interview, oral question-and-answer) are those which are more formally structured.

The Wilcoxon test revealed the following statistically significant difference according to teachers' perceptions of ease of application of these techniques. Before the attendance of the course, oral question-and-answer was considered as the most easy technique. However, after the attendance of the course it was considered as neither the most nor the least easy technique. Thus, Wilcoxon test revealed a relevant statistically significant difference ($Z=3.4$, $p<.001$). The Wilcoxon test revealed also that after the attendance of the course teachers did not consider extended written question as so difficult technique as before the attendance of the course ($Z=2.9$, $p<.01$). Finally, after the attendance of the course, teachers considered the category of oral techniques as less easy ($Z=2.61$, $p<.01$) and the category of written techniques as more easy ($Z=2.8$ $p<.01$). The last, and probably the most important finding, has to do with the well known dilemma that what is easily measured is of dubious educational value. Both before and after the attendance of the course interview and structured observation were considered as the most appropriate but the least easy techniques. Likewise, the direct written question and the unstructured observation were regarded as one of the most easy but least appropriate techniques. It can be argued that, both before and after the attendance of the course, there is a negative correlation between the appropriateness and ease of techniques of assessment.

Perceptions about ways of improving assessment

Kendall's Coefficients of Concordance revealed that teachers agreed among themselves in their ranking of the relative importance of the six ways of improving assessment both before ($W_9=0.48$, $p<.001$), and after the attendance of the course ($W_{10}=0.51$, $p<.001$). Both before and after the attendance of the course teachers considered further training in techniques of assessment and smaller class size as the most important ways of improving assessment, whereas the least important was the existence of another adult in the classroom. Thus, the Wilcoxon test did not reveal any statistically significant difference by comparing teachers' perceptions of improving assessment before and after the attendance of the course.

IV) DISCUSSION

The findings derived from this study can be seen as providing information on the extent to which the professional training may influence perceptions of teachers who work in a highly centralised system. This study can be therefore seen as a case study of a closed system and hence the extent to which it is generalisable to other systems is questionable. However, some more general issues for theory of curriculum change in Mathematics may be also raised.

Howson (1989, p.18) believes that "clear objectives are needed but to be effective they must be objectives accepted by teachers", a view that is the basic focus of my research. It is clear from the questionnaire responses that after the attendance of the course teachers' perceptions of the purposes of Mathematics generally conform to the purposes emphasised in the current curriculum reform in Cyprus. Teachers did not only support the purposes relating to investigative tasks and promoting mathematical knowledge and thinking but also the purpose concerned with the development of pupils ability to talk about Mathematics. After the attendance of the course teachers had a coherent view about active pedagogy, emphasising the value of practical activities, investigative tasks and discussion. Moreover, teachers perceived formative purpose of assessment as more important than the summative. They also considered assessment as a natural part of teaching. Thus, they agreed that pupils' attitudes to Mathematics and pupils' ability to apply their knowledge in unfamiliar situation are important objectives for teachers' assessment. But despite the fact that teachers agreed with policy on curriculum reform in Mathematics, further research is needed to explore the extent to which this policy may influence curriculum practice. Although teachers hold strong ideas favouring active pedagogy and formative assessment, teachers' responses to items on classroom organisation reveal that they did not promote flexible classroom strategies either before or after the attendance of the course.

Since curriculum change is a multidimensional phenomenon, a distinction between changes which affect the deep structures of the curriculum and changes which affect the surface of the curriculum is needed to explore the effect of professional training upon teachers' perceptions. The data about the effect of this course upon teachers' perceptions revealed that after the attendance of the course teachers changed their perceptions of curriculum policy but they did not change their beliefs about their role. For example, their reactions to a school which is in a position to develop its own policy on teaching Mathematics revealed that their perceptions are primarily effected by the system within which their training is operating. Thus, the extent of central control on the curriculum may be a stronger source of influence upon teachers' perceptions of their role in the process of change than training. Nevertheless, the role of training should be seen in terms of Fullan's (1991) argument that change at the individual level is a process whereby individuals alter their ways of thinking and doing. This is not an easy task since people's beliefs are part of a deeply rooted belief system based on perceptions of their role and which extends to social and political concerns. Educational policy in Cyprus has not dramatically used INSET to bring about change and has not been directed at the implementation of the current curriculum reform at school level. Thus, a reform of teacher in-service education is needed so that a link between professional development and curriculum reform will be established.

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Student Motivation and Attitudes in Learning College Mathematics Through Interdisciplinary Courses

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An NSF-sponsored project at Indiana University is developing interdisciplinary, courses in which students learn mathematics in the context of other fields of study (e.g., history, art, criminal justice, economics, speech and hearing, business). Since fall 1996, nine new interdisciplinary courses have been developed and field tested with students and six more are being planned. This paper reports on research on students' motivation to study mathematics and their perceptions of the utility and importance of mathematics when enrolled in these new interdisciplinary courses. .

Traditional mathematics courses, designed and taught exclusively by mathematics faculty, are seldom perceived by students other than mathematics majors as an integral part of their college education. Nor do most students view their study of mathematics as important in preparing them for the challenge of our increasingly technological society. Instead, required college mathematics classes have often been viewed primarily as a set of hurdles to jump over prior to graduation. Much of this is due, no doubt, to the fact that traditional mathematics courses typically do little to help students appreciate the integral role that mathematics plays in the society in general, and, more particularly, in college majors such as arts, humanities, social sciences, or biological and health sciences. As a result, students studying these disciplines often have little motivation to learn mathematics; many see the study of mathematics as little more than the memorization of rules, computational techniques, and equations unrelated to their own personal lives.

In an effort to remedy this unfortunate situation, faculty at Indiana University, USA, under sponsorship of the National Science Foundation for their "Mathematics Throughout the Curriculum Project" (MTCP), have been working to create a new framework for collaboration between mathematics faculty and the faculties of other disciplines, as well as a new culture among undergraduates. The ultimate goal of the project is to revitalize the learning of mathematics among non-mathematics majors. The project seeks (1) to create interdisciplinary courses, developed through collaborative efforts of faculty from mathematics and other disciplines (but eventually to be taught by individual instructors), in which students use previously learned mathematical ideas, learn new mathematics, and acquire new tools through contextualized problem solving, and (2) to change students' attitudes toward mathematics by developing an infrastructure where they are encouraged to see the value of understanding underlying mathematical concepts and to see the relationship of mathematics to other subjects, the real world, and their own ambitions and goals.

As evaluators of the MTCP, our research—since fall semester 1996—has involved collecting data from students, faculty, and administrators about the impact of these new interdisciplinary courses. Our focus in this paper is on our efforts to document progress toward the second of the two primary project objectives—that is, a change in students' motivation and attitude toward mathematics.

Background

Among the many reasons for a reexamination and major overhaul of the current US mathematics instructional system is the significant dropout rate from mathematics study as students proceed through secondary school and college study. For example, of the 3.6 million ninth graders taking mathematics in 1972, fewer than 300,000 survived to take a college freshman mathematics class in 1976, and only 11,000 earned bachelor's degrees in mathematics in 1980 (National Research Council, 1989, 1990). It is clear that many college students do not perceive mathematics as important for their professional development.

Harel and Trgalova (1996) provide a comprehensive review of significant and interesting research investigations and teaching experiences in college level mathematics instruction worldwide. Although their discussion focuses primarily on recent changes in the teaching of calculus and linear algebra, their findings also have significance for the more basic mathematics studied by students in our Indiana University reform effort. In their review, Harel and Trgalova cite Artigue (1995) as suggesting that numerous curricular renovation initiatives have been initiated due to the finding that it is easy to teach college students to compute mechanically, although difficult to teach them underlying concepts. Indeed, this is exactly the point of much of the current US reform at all grade levels. For example, the *Curriculum and Evaluation Standards for School Mathematics,* published by the (US) National Council of Teachers of Mathematics (1989), calls for a "shift in emphasis from a curriculum dominated by memorization of isolated facts and procedures and by proficiency with paper-and-pencil skills to one that emphasizes conceptual understandings, multiple representations and connections, mathematical modeling, and mathematical problem solving" (p. 125). Though the NCTM Standards were written as recommendations for school mathematics, they seem to apply equally well to the situation we face with college mathematics.

Some recent reform efforts at the college level in the US have emphasized encouraging students to work in small groups to construct mathematical ideas through projects and applications. An example is the *Calculus in Context Project* (O'Shea & Senechal, 1992). In general, however, there has relatively little attention given to reform of precalculus mathematics instruction (which is the focus of the current Indiana project). Much recent curriculum reform at the college level has been technology driven. Students using *Mathematica, Derive,* or *Maple* have been reported to exhibit greater willingness to solve problems (Davis, 1992). It is also reported that weaker students often are better able to succeed with the help of technology, and thereby come to recognize that mathematics is not just for their more able classmates (Wimbish, 1992).

Overview of Selected MTCP Courses

To provide readers with some sense of the types of courses being developed by the MTCP, we offer the following brief course descriptions.

"Techniques of Data Analysis," a course required of all criminal justice majors, was developed and team-taught by a mathematician and a criminal justice professor. The course was conducted in a computer lab and enrolled 22 students, ranging from sophomores to seniors, in fall 1997. Prerequisite mathematics was a course in either precalculus or finite mathematics. This new course required students to apply mathematical concepts and statistical procedures to specific criminology-related data sets. The course goals included learning about the mathematics underlying statistical analysis of social sciences data and developing skills in using a computerized statistical software package. Sophisticated statistical concepts were explained during lectures. The course included group projects and hands-on computer work. An example of a course project was the use of contingency tables in analyzing a large data base drawn from a citizen survey.

"Mathematics in Action: Social and Industrial Problems," collaboratively developed and taught by a mathematics professor and a business administration professor, enrolled 24 students from a wide range of majors, who ranged from freshmen to juniors. The only prerequisite was precalculus mathematics. The course aimed to teach students the content of the traditional (required) finite mathematics class through contextualized problem solving in the realm of business. The major highlight of the course was the use of real-world industrial projects proposed by local businesses and industries. In addition to the industry project, the instructors also used simplified versions of actual real world problems taken from professional journals and books as well as a diverse set of applications of mathematical concepts. Graphing calculators and computers were used as technological tools in the course. Two sample projects were (1) developing a transportation-shipping schedule (given supply and delivery points for shipping, the transportation model needed to look for that set of origin/destination points that would minimize transportation costs), and (2) developing a products-defect model by using statistical analysis to calculate the probability and the expected time during a warranty period until a given product part would become defective (in this project, the actual problem was to help a school district decide how long they should keep their school buses and other vehicles before buying new ones).

"Mathematics and Art," taught by a mathematics professor who is also an artist enrolled 15 students (from sophomores to seniors). Precalculus was the only as prerequisite. The class was designed to explore connections between mathematics and art through history, as well as through recent developments in art, mathematics, and computer graphics, and its goals were to create an intrinsic motivation to learn mathematics through art as well as to emphasize the connection between mathematics and art.. Students examined art concepts such as symmetry, perspective, and landscapes as well as mathematical topics such as Euclidean geometry, analytic geometry, and iterative techniques. Although the class met in a regular classroom, students also used a variety of graphics software in a computer lab. Examples of course projects were use of fractals to reproduce both a branch of a tree and the texture of an old vase.

"Mathematical Foundations of Speech and Hearing Sciences," enrolled 18 students (sophomores to graduate students) in fall 1997. It was designed for students majoring in Speech and Hearing Sciences and related fields such as music, psychology, and cognitive science, but was open to all students. Prerequisites were either one semester of calculus or finite mathematics. The course aimed to develop a solid foundation in mathematical concepts underlying the speech and hearing sciences and to develop the ability to apply these concepts to practical and clinical problems. Anticipating that students would solidify some previously learned mathematics and also learn new content, the instructors focused on the major mathematics topics encountered in upper level speech courses: trigonometry, Fourier analysis, finite mathematics random processes, and statistical decision theory. Two days each week the focus was on mathematical concepts and related speech and hearing problems. One day each week students did hands-on lab work related to real projects, using graphing calculators and computer spreadsheets.

"A Statistical Study of History," taught by a historian and a mathematician, enrolled 21 freshmen in fall 1997 and required only high school algebra as prerequisite. Goals of this course included helping students develop the art of understanding history and their contemporary world through a reasoned quantitative approach, and helping them to employ this approach to validate or refute standard interpretations of nineteenth century Indiana history. Using appropriate statistical methods, hand-held calculators, spreadsheets, and statistical software, students analyzed data from the 1860 Indiana Federal Census database. In doing so, they learned about statistical topics varying from descriptive statistics to probability models and statistical inferences. An example of a student project was comparison and analysis of statistics describing minority households with a single parent during various time periods from 1920 to the present.

Methodology

As evaluators for the MTCP we have used a variety of measures and techniques for evaluating progress toward major project objectives. To document progress toward objective #1—creating interdisciplinary courses—we have:

- examined course syllabi, samples of assignments, tests and other assessments, samples of student work, and other documents to determine if the courses meet project goals of: (a) requiring minimal mathematical prerequisites, (b) being problem/project driven, (c) involving student activity and group work, (d) emphasizing communication.

- determined the extent to which student learning outcomes in mathematics are (a) specified in the syllabi, (b) taught in the course, and (c) appropriately evaluated.
- observed in classrooms, with attention to mathematics content, teaching methods, and assessment methods, to triangulate other measures of course success.
- examined uses made of technology (calculators, computers, videos, Internet, etc.)

To document progress toward objective #2—changing student attitudes—we have:

- administered surveys to students at the beginning and end of each semester to obtain measures of student expectations, attitudes, beliefs, and satisfaction.
- interviewed selected subsets of students and interviewed all of the course instructors.
- observed in classrooms, with particular attention to behaviors that might provide evidence about anxieties, motivation, beliefs, and attitudes toward mathematics.

Progress Toward Project Goals

In general, our data collection has documented significant progress toward project goals. Nine new interdisciplinary courses have been developed and field tested (the five described above, plus a biology course, a physics course, and two courses in economics), and both student and instructor reaction has generally been quite positive. Our interviews, classroom observations, and document analysis efforts have yielded promising evidence of success. Unfortunately, our paper-and-pencil attitude survey was, at least initially, less informative

At the time of writing this paper, we have student questionnaire data from each of the courses field tested during spring semester 1997, though we also expect to have questionnaire data from fall 1997 and spring 1998 by the time of the PME22 meeting in summer 1998. Unfortunately, the questionnaire used to gather data on changes in student attitudes during spring 1997 did not measure any significant differences from the beginning to the end of the semester. We believe this is probably due more to the insensitivity of the questionnaire than to an absence of change, particularly because interviews with students and classroom observations provided a very different look at students' attitude changes. In summer 1997, the student questionnaire was completely revised, so we expect more reliable results from data collected in fall 1997 and spring 1998.

Data sources other than the paper and pencil attitude survey seem to offer clear evidence that many students' views of mathematics have been changed by taking one of the new interdisciplinary courses. These data sources include individual student interviews, written answers to open ended questions included on end-of-semester evaluation forms, and interviews with course instructors and others involved with the courses. In the paragraphs that follow, we focus, due to space constraints, on findings related to just one course: the business administration course. (More extensive information and analysis about this and other courses will be available at the PME22 meeting.)

"Mathematics in Action: Social and Industrial Problems" was a project-driven finite mathematics class that promoted student learning of mathematics through problem solving. It was a special section of a course typically offered in multiple sections and required for many majors. This section covered the same content as the usual course, including descriptive statistics, probability, counting, systems of linear equations, and linear programming. Students from a variety of different disciplines enrolled. They learned the mathematics, used it in problems related to applications, and discussed those problems during class. The course included student activities and group work and also emphasized communication.

Students applied the mathematical concepts they were learning by using them as tools in team projects drawn from local industry. During one of our class observations, we saw the students spend more than half the class period debating and discussing such applications. The students asked questions of each other and of the instructors, who helped students connect the underlying mathematics to the applications at hand. Both the mathematician and the business administration professor were actively involved in these exchanges. While one was keen about the mathematics, the other was more conversant with the applied problems.

Many of these students expressed a desire to take more mathematics classes, particularly if those classes could be like this one. One student claimed that she would miss coming to this class when it was over. Several students said that even though the class met very early in the morning (8:00 a.m.) they were reluctant to be absent. This was evident during the class observations. About 10 minutes before the class started, there were typically students in the classroom, already discussing problems with other students or with the instructors.

During interviews, students claimed that now they understood the utility of mathematics and its power. They were surprised to see how much mathematics they actually use in their real life. The course exposed them to the statistical analysis programs build into spreadsheets such as Excel and Lotus and the use of calculators. They expressed surprise at how much those programs could do. Also, the real life problems they encountered in their industrial projects seemed to be eye-openers for many students. Some mentioned that they cannot now look at a newspaper without seeing (and understanding) many uses of the mathematics they learned in the class. Some students who had already taken other college mathematics, such as calculus, said they now see much more clearly the uses of that mathematics and the reasons behind using it. A senior student was assigned to assist students in this class. She worked primarily in the computer lab helping students to learn Excel and giving assistance with course projects. Additionally, she worked in the tutoring lab at the university, where she helped students with mathematics from various classes. According to her, students in this class had much better understanding of the concepts learned in this class than their peers who were going through the parallel, regular finite mathematics class.

From looking at students' attendance, participation, and assignments, it seems clear that students were very enthusiastic about this class. At the beginning there were some students who came to the first class and then dropped out due to the anticipated heavy workload. Of those who persisted in the class, most said that the class involved a lot of work, but that they enjoyed it nonetheless. Many of the students in this class participated in workshops and labs that were not required and that were held outside the class period. They also met with their project groups outside of school. One student interviewed said that her group had to meet 7:00 a.m. in a nearby small town because that was the only time and place all group members could get together. From class observations as well as student interviews, it was evident that these students came to enjoy mathematics during this class. They did not fear mathematics and they saw the usefulness of it.

There was a diverse group of students in this class (business and economics majors, liberal arts and science majors, and preservice elementary teachers). Therefore, when they were asked to select an industrial project from among a selection of projects, the topics that they chose were quite diverse (and often related to their own personal areas of interest or experience). Students particularly noted that they learned a lot from each other in doing these projects, because their fellow students had expertise in a diversity of areas. They all acknowledged the projects as a very constructive and enjoyable experience.

Developing this new course was labor intensive for the instructors. Both instructors were present everyday in the classroom, and both were involved in all class discussions. Students mentioned during interviews how positive it was that the instructors offered such diverse perspectives because they came from different academic disciplines. Other labor intensive aspects of the course were the extra workshops and labs that the instructors arranged and monitored, and the time and effort they expended in contacting local industries to organize the industrial projects. They also visited all the industrial project sites and attended initial project meetings of the student investigators. During interviews, the instructors admitted that developing and teaching this course involved a lot more work than usual. However they also expressed a sincere sense of satisfaction in teaching the class. One instructor mentioned that he felt closer to the students in this class than in any other class he had taught before.

Looking Back and Looking Ahead

We are convinced that learning through applications helps students develop an appreciation for mathematics and a motivation to learn that is often lacking in traditional mathematics courses. Similarly, we have noted that the instructors in these innovative courses report being more engaged and more positive about both their subject matter and their interactions with students than during their typical teaching. At this early stage in the MTCP's life, however, we have no way of knowing how long this enthusiasm on the part of students and faculty can be sustained. What will happen when faculty are no longer team-teaching? Once these courses are institutionalized, how successfully will a single instructor be able to offer the diversity of background and expertise presently provided by the MTCP interdisciplinary teams? Teaching applications-oriented, project-based courses requires considerable extra effort and commitment. How long can faculty interest and energy be sustained? Finally, successfully disseminating the courses developed by this project to other campuses, colleges, and universities remains the project's biggest hurdle. How effective will the courses be when they are taught by instructors who had no involvement with their interdisciplinary development? Changing the motivation and attitude of the students enrolled in these experimental courses has not been terribly difficult; offering similar opportunities to a whole new generation of students is the ultimate challenge.

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Algebra: Meaning through Modelling

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A modified version of classical mathematical modelling was used to introduce middle school students to algebraic symbols and to enhance their perception of the usefulness of algebra in solving practical problems. After the first year of a three-course algebra sequence, grade 7 students, ages 11-12, surpassed students two years beyond them in the curriculum in quantitative reasoning and elementary modelling.

Background

In the United States, there has been considerable debate over algebra and pre-algebra courses. Many students experience difficulty when algebra is taught as a series of isolated skills (simplify, evaluate, factor, etc.) and the traditional curriculum is criticized for its failure to address the content that best reflects societal uses and students' future vocational needs (Steen, 1992; NCTM, 1992; Chambers, 1994). Although the Standards (NCTM, 1989) proposed a different conception of algebra, they provided only broad guidelines for creating change. Without the benefit of a solid research base that addresses essential content and effective pedagogy, many school districts have responded to mandates that all students study algebra (Mathematics Education Trust, 1990; The College Board, 1990), by providing access to algebra courses, but not necessarily access to critical algebraic ideas (Silver, 1997).

At the heart of the controversy over algebra lies the question about what algebra is and should be. Generalizing and formalizing patterns and the syntactic manipulation of symbols are viewed as kernels of the subject (Kaput, 1995), but mathematical content such as structures, variables and functions, relations and joint variation, and technological innovations, also help to define the nature of algebra. Rather than presenting algebra as an isolated topic of study, there is a growing consensus that it should be located within a larger, longitudinally coherent mathematics curriculum that focuses on understanding quantitative relationships and the mathematization of authentic experiences (Kaput, 1993; NCTM, 1992).

Model formulation, the first stage of the modelling process, involves discerning important variables and parameters that affect a given problematic situation, making assumptions to clarify and simplify the situation, making a conjecture about the relationships among important quantities, and translating the relationships into a mathematical statement. The teaching experiment reported in this paper was based on the assumption that model formulation provides a powerful forum for helping students to think about quantities and their relationships. Focusing on the reasoning that precedes the production of a symbolic equation provides students the opportunity to see algebra as an activity; to appreciate its utility; to see familiar situations as a source of meaning for formal mathematical symbols; to develop a systematic approach to analyzing and communicating the underlying structure in a situation; and to build a firm foundation for more abstract reasoning.

The Teaching Experiment

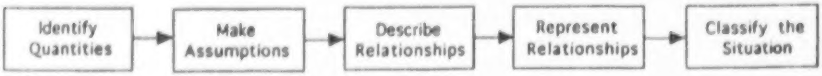
Responding to the need for change, a small, mid-western school district in the United States revised its algebra program. After many years of tracking students in the middle school, the more capable students were being pushed farther and farther into the high school mathematics curriculum. Students were covering algebra two to three years before they would normally encounter the subject in the first year of high school, but learning only the rote manipulation of symbols. They had little understanding of how, when, or why the symbols might be used. As a result, the middle and high school algebra teachers cooperatively devised a three-course pre-algebra and algebra sequence for students in grades 7 through 9, which allowed students ample time to deeply investigate quantitative relationships and to mathematize everyday situations. Student progress was documented during the first year of the first course, for the purpose of demonstrating that modelling could help the students to find meaning and utility in algebraic symbolism.

Subjects

One class (N= 16; 9 males, 7 females) of seventh grade middle school students who were identified as mathematically talented students, participated in pre-algebra 1, the first course of the algebra sequence. Teachers identified these students based upon their performance in whole number, decimal, and fraction computation at the end of sixth grade.

Pedagogy

To help students develop a systematic way to analyze the relationships among quantities in a problem, the model formulation stage of the mathematical modeling process was elaborated in the following five-step process.



For the first six weeks of the course, students used the diagram to guide their analysis of simple statements such as the following:

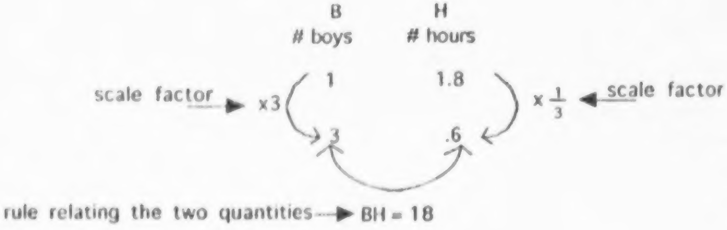
> If 1 boy can mow the lawn in 1.8 hours, 3 boys could probably mow the lawn in .6 hours.

Identifying quantities meant naming the quantities that were related, using both a number and a unit of measurement. That is, rather than merely listing "boys" and "time" as the significant quantities in this situation, students labeled the quantities "number of boys" and "time in hours."

Defining the situation meant making explicit the assumptions needed to keep a situation sensible and tractable. For this situation, it was assumed that both boys were willing and able to do about the same amount of work in one hour. Students came to appreciate that in mathematics, one always sets out the conditions under which a discussion or proof evolves and that any conjectures or results are subject to those conditions.

Describing meant making a verbal statement about how the quantities were related. For example, "If you have more people doing a job, it should take less time to get it finished than if only one person is doing it." or "As the number of people goes up, the time to do the job goes down."

Representing meant using arrow notation to describe the relationship between the significant quantities in the situation, and later, representing the relationships in quantity diagrams. If B represents the number of boys and T represents the time it takes to mow the lawn in hours, the quantities change in the following way: B↑ T↓. As students realized that not all situations within the ↑↑ or ↓↓ or ↑ ↓ categories were the same, further analysis was needed. They learned to create quantity diagrams, such as the following:



Classifying meant associating the situation under consideration with others whose relationships are similar, and refining categories when necessary. For example, after comparing the lawn mowing problem to the dying cell problem below, students were able to see that all ↑ ↓ statements were not the same and that it was necessary to subdivide the category.

> After the first 1.5 hours, the cell count was down to 40. Thereafter, every three hours, half of the remaining cells died.

As categories were refined, more conventional algebraic language was used to describe the relationships (e.g., proportional, inversely proportional, exponential).

Students kept files of the problems they had analyzed. Problems having the same structure were filed together; new categories were added or existing categories were refined when necessary. In time, students identified structurally similar relationships, used the language of algebra to describe their chief characteristics, and associated them with situations they had analyzed. Classroom activities ranged from the quantitative analysis of simple statements in the beginning of the year, to structured modelling in the middle of the year, to more open modeling at the end of the year. Students analyzed and represented quantitative relationships in verbal, pictorial, graphical and tabular form.

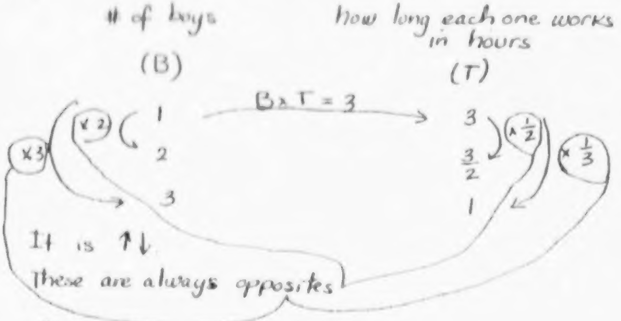
Results

Although all of the students were mathematically talented, they were accustomed to listening and practicing in their mathematics classes, and their adaptation to reasoning and writing about mathematics was accomplished with varying degrees of difficulty. To indicate the range of students' performance in the course, periodic work samples from Nicki (female), the strongest student in the class, and Dan (male), who made the least progress throughout the year, are given below.

November: Mowing the Lawn

If one boy can mow a lawn by himself in 3 hours, then how long should it take when his friend helps him? What happens if more friends decide to help?

Nicki I think we have a new relationship. Here is my diagram, then I will show how I got it.



And there is a rule B × T = 3

Other people think it works like this: every time you add another person, the work gets cut in half. But here is the right way to do it.

# of boys	how much of the lawn each one does	how long each one works together in hours.	
1	ALL	3	
2	$\frac{1}{2}$	$\frac{1}{2}$ of 3	$\frac{3}{2}$ or $1\frac{1}{2}$
3	$\frac{1}{3}$	$\frac{1}{3}$ of 3	1
4	$\frac{1}{4}$	$\frac{1}{4}$ of 3	$\frac{3}{4}$
5	$\frac{1}{5}$	$\frac{1}{5}$ of 3	$\frac{3}{5}$
6	$\frac{1}{6}$	$\frac{1}{6}$ of 3	$\frac{3}{6}$ or $\frac{1}{2}$

Dan

# people   hours to cut lawn

$$\begin{tikzcd} 1 \arrow[d, bend right, "\times 2"'] \arrow[r, "\text{no rule}"] & 3 \arrow[d, bend left, "\div 2"] \\ 2 & 3 \div 2 = \frac{3}{1} \cdot \frac{1}{2} = \frac{3}{2} = 1.5 \end{tikzcd}$$

P↑ H↓

When the number of people doubles, the hours get cut in half.

April: Spreading the AIDS Virus.

Using the dominoes you have been given, show three different models that might be used to describe the growth of the AIDS virus.

Dan

Pattern A each dominoe hits one dominoe each time
B each dominoe hits two dominoes each time
C dominoes in △ shape 1 hits 2, 2 dominoes hit 3, 3 hit 4

A

number of steps	number of aids cases
1	1
2	2
3	3

B

1	1
2	3
3	7
4	15

C

1	1
2	3
3	6
4	10
5	15

B took 4 steps to get 15 case. C took 5 and A took 15. So B is increasing faster.

Nicki ① 1 → 1 → 1 → 1 → etc.

#1 assumes that every contact causes 1 new aids person

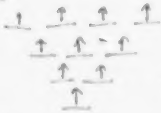
rule p = c

C = # of contacts

p = # of aids people

# of contacts	# of aids people	
0	1	1
1	2	1
2	3	1
3	4	

②



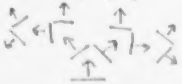
each new aids person contacts 2 other people

#2 assumes when 2 aids people have contact with the same person, only 1 new case of aids happens

rule $p = \frac{1}{2} \cdot (c+1) \cdot (c+2)$

# of contacts		# of aids people	
0		1	2
1		3	3
2	÷2	6	4
3	÷2	10	5
4	÷2	15	6
5		21	

③



#3 assumes aids people contact 2 new people each time

rule $p = 1 + 2^1 + 2^2 + 2^3 + \ldots + 2^c$

# of contacts	# of aids people	
0	1	$2 = 2^1$
1	3	$4 = 2^2$
2	7	$8 = 2^3$
3	15	$16 = 2^4$
4	31	$32 = 2^5$
5	63	$64 = 2^6$
6	127	

In #1 the increase is constant

In #2 the increase is c+1

In #3 the increase is 2^c

#2 increases faster than #1, but #3 is fastest. It is exponential growth. It starts slow but grows fast.

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June: Supermarket Check-Out

Create a model for deciding which check-out line to join in a supermarket.

Dan P = number of people
I = number of items

$$\overset{1}{P_1 I_1} \qquad \overset{2}{P_2 I_2} \qquad \overset{3}{P_3 I_3}$$

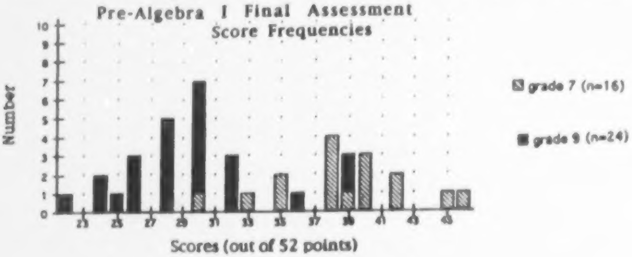
Look at each asile and see which number is bigger.

Nicki: p = # of people in the line
n = # of items in their cart
t = time to scan 1 thing = 3 seconds
x = time to pay = 20 seconds
time to check out is $3n + x$ seconds
for p people, it will take $p \cdot (3n + x)$ seconds
do that for each line and pick the one that takes the smallest amount of time

Final Assessment

The final examination included such tasks as finding the nth term of a given sequence, figuring out which of two ski slopes was steeper, figuring out how much carpeting was needed for a staircase of any number of steps when rise and run were given, and dividing $ M between two people so that one gets x times the other. Although it was not the intention of this study to compare the content, pedagogy, or results with any other class, one high school algebra teacher gave his algebra I students seventh grade examination. Numerical scores were assigned to each of 13 problems using a 4-point rubric, subject to the consensus of all of the middle and high school algebra teachers. The performances of both classes are shown below.

During the year, students participated in weekly interviews to document the development their growth in thinking. Case studies (to be reported elsewhere) show that the seventh grade students had gone well beyond surface-level investigations to seek out deeper structure. Students demonstrated that they understood the difference between variables and parameters (e.g., see Nicki's June work sample) and were able to move beyond particular solutions to find generalized solutions. They were able to distinguish different uses for a variable: as an unknown, as an argument, and as a pattern generalizer. They understood rates of change. The interviews also documented students' developing knowledge about relations and functions, the use of relative thinking, and proportional reasoning.



Discussion

Had these seventh grade students taken the examination given to the ninth grade algebra I students, they probably would not have known the quadratic formula, how to factor, or how to solve all of the equations and inequalities. Nevertheless, by the end of their first year, the algebraic language, concepts, meanings, and reasoning that they had acquired, were well-connected, practical, and grounded in familiar situations. They still had two years to perfect the tools of algebra and to learn symbol manipulation. On the other hand, the ninth grade students who could not reason quantitatively nor solve simple modelling problems had finished their formal study of algebra with lots of tools and no sense of when to use them.

This was a pioneering attempt to by a single school district to operationalize algebraic reasoning and to demonstrate one way in which the ideas of algebra might be incorporated into a longitudinal curriculum. Other such attempts are badly needed to help build a vision of how useful algebra can be learned with understanding.

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AN ALTERNATIVE ASSESSMENT: THE GUTIÉRREZ, JAIME AND FORTUNY TECHNIQUE

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In the early 80s Mayberry (1981) developed a diagnostic instrument to be used to assess the van Hiele levels of pre-service teachers. A replication of Mayberry's work was undertaken at the University of New England. As part of the analysis, the students' responses were re-assessed using the technique developed by Gutiérrez, Jaime and Fortuny. This paper presents an evaluation of this assessment method.

The ability to instruct students at their level of understanding is dependent, in part, on the teacher being able to assess students' levels of understanding. In order to make this assessment, there needs to be available a reliable diagnostic instrument. In the early 80s Mayberry (1981) in her work with pre-service primary teachers, developed such a diagnostic instrument that could be used in an interview situation. Mayberry's test items and method of evaluation are based on the key assumption that the van Hiele levels are discontinuous (Mayberry 1981, p.22). This led Mayberry to design each item to test for understanding of a specific van Hiele level, the response being assessed on whether it reflects that level of thinking. There is no grading of the degree of difficulty of the Mayberry items within a level, nor of the depth of understanding of the level displayed in a response.

An alternative paradigm for the evaluation of the acquisition of van Hiele levels by students has been presented by Gutiérrez, Jaime and Fortuny (1991). In contrast to Mayberry, they have based their research on the idea that the van Hiele levels are not discrete, rather that they are of a more dynamic nature, that they are continuous rather than static (Pegg, 1992, p.25). Their theory (Gutiérrez, Jaime and Fortuny 1991, p.237) is based on observations that, when answering questions, although most students show a dominant level of thinking, a response frequently displays some reflection typical of another level. This paper presents an evaluation of their coding system in determining van Hiele levels displayed by the students in their responses to the Mayberry items. Before presenting the evaluation of the assessment method of Gutiérrez *et al*, a brief background to the important ideas underpinning their work is presented.

Background

The van Hiele Theory

In the 1950s, Pierre van Hiele and Dina van Hiele-Geldof completed companion PhDs which evolved from the difficulties they had experienced as teachers of

Geometry in secondary schools. Whereas Dina van Hiele-Geldof explored the teaching phases necessary in order to assist students to move from one level of understanding to the next, Pierre van Hiele's work developed the theory involving five levels of insight. A brief description of the first four van Hiele levels, those commonly displayed by secondary students and most relevant to this study, is given:

Level 1 Perception is visual only. A figure is seen as a total entity and as a specific shape. Properties play no explicit part in the recognition of the shape.

Level 2 The figure is now identified by its geometric properties rather than by its overall shape. However, the properties are seen in isolation.

Level 3 The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised.

Level 4 Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions are used with understanding.

The van Hieles saw their levels as forming a hierarchy of growth. A student can only achieve understanding at a level if he/she has mastered the previous level. They also saw the levels as discontinuous, i.e., students do not move through the levels smoothly (van Hiele, 1986).

The research of Gutiérrez, Jaime and Fortuny

The method of evaluation developed by Gutiérrez, Jaime and Fortuny, based on the premise that the levels are continuous, results in a qualitative assessment of a student's degree of reasoning in each of the four levels. Gutiérrez, Jaime and Fortuny (1991, pp.238-239) maintain that initially students are not aware of the new, higher level of thinking. They have no acquisition of that level. As they become aware of the new level, an attempt to work at the level is made and a low degree of acquisition is acquired. Continual growth in awareness is shown in an increasing degree of thinking by the students at this level, through an intermediate degree of acquisition, a high degree, until they have a complete acquisition of the thinking at that level.

Several steps are necessary in evaluating a student's van Hiele level(s) using the method of Gutiérrez *et al.* First, in considering a response, the highest level of reflection displayed in the response needs to be determined in order to give the student full credit for the understanding displayed. In making this decision, it can be necessary to consider the response in conjunction with the student's other answers (p.239). For example, a response which appears to be close to the necessary and sufficient conditions sought in Mayberry Item 24 (next page) can be an attempt at expressing minimum conditions (Level 4), or it can be a statement of the few properties known by the student for that topic (Level 2). Consideration of the student's other responses is necessary to determine which is the correct level.

Item 24

Circle the smallest combination of the following which guarantees a figure to be a square.

a. It is a parallelogram.
b. It is a rectangle.
c. It has right angles.
d. Opposite sides are parallel.
e. Adjacent sides are equal in length.
f. Opposite sides are equal in length.

It is not always necessary to consider other statements or responses. Many students in their responses consistently show a dominant level of thinking. With such students, their statements constantly re-confirm their most common level of reasoning. However, the assessor needs to be aware of students who are beginning to explore a higher level for some aspect of a concept and may give a better than expected response if the question is focusing on that aspect or characteristic. Conversely, students who have been attempting to work at a new level of reasoning sometimes, in a response, revert to a lower level which is more familiar to them.

Having decided on a level, the response is now assigned one of eight types of answer. This categorisation depends on the degree of mathematical accuracy, and on how complete the solution to the question is. Gutiérrez *et al* (pp.239-240) explained "To determine which type an answer belongs to, it is necessary to consider it from the point of view of the van Hiele level it reflects, since an answer can be adequate according to the criteria of a given thinking level but not valid according to the criteria of a higher level."

These two steps result in an answer being assigned a vector (*l, t*), which shows the highest level (*l*) the answer reflects, together with the type (*t*) of answer according to its completeness and correctness. The responses are then quantified according to each vector, and the student's degree of acquisition of each van Hiele level determined by calculating the arithmetic mean of the values of the student's answers to those items that could have been answered at that level. If a response has been given at Level *(n)* when the question could have been answered at Level *(n+1)*, a zero score is given for Level *(n+1)*. However, if a response has been given at Level *(n)* and the question could also have been answered at Level *(n-1)*, a score of 100 is allocated to Level *(n-1)*, since a response at Level *(n)* implies complete acquisition of Level *(n-1)* (p.246). Finally, the student is assigned a qualitative degree of acquisition when the arithmetic mean is converted to a subjective interpretation of No acquisition (0-15%), Low (15-40%), Intermediate (40-60%), High (60-85%) or Complete (85-100%) acquisition.

In formulating their alternative paradigm for the evaluation of the acquisition of van Hiele levels by students, Gutiérrez *et al* (p.239) started with some assumptions. These are:

- that it is more important to observe the students' type of reasoning than their ability to solve certain problems correctly in a set time,
- that a partially correct (or even a totally incorrect) answer may also afford information, and
- that an incorrect answer, when considered in conjunction with other answers, may give more than a negligible amount of information.

Design

In order to consider Mayberry's work in an Australian context, a detailed study of the geometric understanding of 60 first-year primary-teacher trainees was carried out at the University of New England (Lawrie, in press). The study aimed, in part, to provide a written test based on the Mayberry interview schedule. Follow-up interviews were conducted with students to validate the levels of thinking as determined in the written test. Mayberry's items covered seven geometric concepts, namely, square, right triangle, isosceles triangle, circle, parallel line, congruency, and similarity. Level 5 items were omitted, hence the written test assessed van Hiele Levels 1 to 4. Conversion of the Mayberry items to a written test involved some modification of the wording to ensure that the intention of each question was clear. A preliminary study validated the reliability of the written questions. In her study, Mayberry assessed the response to every question part equally, whether the question required simple yes/no answer, or whether it required a complex explanation. A criterion was set for each concept and level (ranging from 50% to 100%). If a student has given sufficient correct answers to reach Mayberry's criterion, the student is credited with having mastered that van Hiele level.

The students' responses were re-assessed using the method developed by Gutiérrez *et al* (1991). To allow for the unequal expectations for some of the Mayberry question parts, e.g., the difference between the expectations of a yes/no type of question compared to a question requiring a complex explanation, groups of question parts within one item in the Mayberry test were assessed as one complete item. In addition, the initial data were further analysed to substantiate many of the general observations, using the QUEST program (Adams & Khoo 1993) which is based on the Rasch measurement theory.

Results

Table 1 shows the number of students demonstrating reasoning for each van Hiele level and for each of the seven concepts, when assessed by the method developed by Gutiérrez *et al.*

Table 1

The Degree of Acquisition Attained by Students for each van Hiele Level

		Degree of Acquisition				
Concept	van Hiele Level	No acquisition	Low	Intermediate	High	Complete
Square n=61	1	0	0	0	0	61
	2	0	2	7	24	28
	3	37	9	13	2	0
	4	59	2	0	0	0
Right triangle n=31	1	0	1	0	1	29
	2	0	2	2	13	14
	3	14	10	2	4	1
	4	30	1	0	0	0
Isosceles triangle n=30	1	0	0	2	0	28
	2	0	2	1	8	19
	3	15	6	5	4	0
	4	29	1	0	0	0
Circle n=31	1	0	0	0	0	31
	2	0	4	0	11	16
	3	17	8	3	2	1
	4	29	2	0	0	0
Parallel lines n=30	1	0	0	1	1	28
	2	0	2	3	7	18
	3	23	6	1	0	0
	4	30	0	0	0	0
Congruency n=31	1	0	0	0	7	24
	2	2	2	5	10	12
	3	16	7	5	3	0
	4	27	4	0	0	0
Similarity n=30	1	0	0	1	0	29
	2	0	5	4	3	18
	3	18	7	5	0	0
	4	26	3	0	0	0

*Similarity Level 4, n = 29

The results agree with the hierarchical structure of the van Hiele levels, and show that the higher the level, the lower the degree of acquisition. Overall, the results confirm the findings of the Mayberry assessment, that the majority of students have mastery of Levels 1 and 2, but little or no understanding of the higher levels, i.e., most students were comfortable recognising concepts, and listing the associated properties, but did not understand the relationships between the properties.

A comparison of the results with those resulting from the Mayberry assessment shows that a high or complete degree of acquisition in the Gutiérrez *et al* results corresponds with mastery of a level in the Mayberry evaluation, while a low or no acquisition corresponds with failure to reach Mayberry's criterion. Results showing an intermediate degree of acquisition (40 to 60) tend to correspond with scores close to the Mayberry criteria, some failing to reach the criterion, others achieving mastery of the level. For example, of the eight students assessed as having mastery of Level 3 for the square when assessed by the Mayberry method, two were shown to have a high degree of mastery, and six an intermediate degree of mastery of the level in the assessment using the method of Gutiérrez *et al*. The remaining seven students assessed as having intermediate mastery all failed to achieve Mayberry's criterion for the concept and level. Hence, the Gutiérrez *et al* method of assessment, in taking into account all responses, whether complete and/or correct, makes a more realistic evaluation, giving a more accurate measure of a student's degree of understanding of geometry. The Rasch analysis supported the theoretical background, and also the results determined using the Gutiérrez, Jaime and Fortuny method of assessment. An increase in the mean difficulty thresholds for the questions set by Mayberry for each van Hiele level agreed with the hierarchical nature of the van Hiele levels, whilst an overlap of the difficulty thresholds for questions testing for consecutive levels supported the notion that the levels are not discrete. The analysis showed also a greater degree of difference between the difficulty thresholds of questions for Levels 2 and 3 than between the thresholds of other levels, indicating that the larger proportion of students were able to demonstrate Level 2 knowledge of properties, but unable to reason with the properties relationally (Level 3).

In conclusion, the alternative paradigm described by Gutiérrez *et al*, because it measures a student's capacity to use each one of the van Hiele levels in every statement made, results in a more flexible interpretation of the reasoning of the student. In particular:

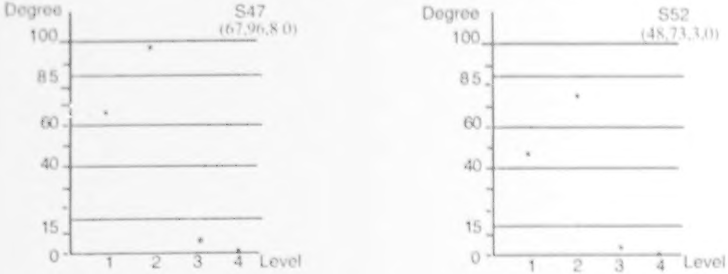
1. a student can be shown to be developing in two consecutive levels of reasoning at the same time,
2. the incorrect assignation of a level to a question is of minimal significance,

3. the effect of unequal distribution of questions across levels in minimised,
4. incorrect assessment resulting from 'lucky' guesses such as in true/false questions, from weak, and from misinterpreted questions is minimised, and
5. inequalities associated with success criteria as in the Mayberry assessment method, are eliminated.

Unusual Behaviour Patterns

An inspection of the quantitative results obtained in the re-evaluation of the students' responses, using the method of Gutiérrez *et al*, reveals that not all results agreed with the hierarchical structure of the van Hiele levels. In twenty-six (2.7%) of the nine hundred and seventy-six assessments, the degree of acquisition of Level *(n)* is not less than the degree of acquisition of Level *(n-1)*. In every case, the non-hierarchical behaviour occurred between Levels 1 and 2, the degree of acquisition of Level 1 measuring below that for Level 2. However, many of the pattern errors in these quantitative results are considered to be trivial, the value of Level 1 being less than 10 points below the value of Level 2. The results of two students (S47 and S52) illustrate cases in which neither the qualitative nor the quantitative assessment fit the hierarchical pattern. S47 was assessed as showing a high degree of acquisition of Level 1, complete acquisition of Level 2 and no acquisition of Levels 3 and 4, whilst S52 was assessed as showing an intermediate degree of acquisition of Level 1, a high acquisition of Level 2 and no acquisition of Levels 3 and 4. The two patterns of unusual behaviour are graphed below.

Graphs of Unusual Behaviour Patterns



The occurrence of these patterns in which the degree of acquisition of Level 1 is lower than it should be, is suggestive of two factors; (a) that the Mayberry questions designed to measure Level 1 are not always clear in their intention, and/or (b) that the criteria adopted to measure the acquisition of levels using the alternative paradigm of Gutiérrez *et al* may be more suited to responses demonstrating reasoning as with Levels 2, 3 and 4 rather than to responses dependant on visual identification (Level 1). The items showing least degree of fit in the Rasch analysis were mostly Level 1 items, indicating that the better students may have had difficulty in interpreting the thrust of the Level 1 questions.

Conclusion

The assessment method developed by Gutiérrez *et al*, in evaluating the degree to which understanding of each level is expressed in every response, provides a realistic mechanism for measuring a student's understanding of geometry. It also provides insight into the quality of a question. However, the alternative paradigm, as developed by Gutiérrez *et al*, also needs further investigation and refinement.

1. Questions at van Hiele Level 1 are not always assessed accurately. Is there a limitation in the ability of the coding system of Gutiérrez *et al.* to evaluate visual recognition?
2. The automatic allocation of a credit of 100 for Level *(n-1)* for an attempt at Level *n* does not always seem justified and possibly presents a contradiction to other aspects of their coding system.
3. The coding system is very time-consuming. Whereas it gives a much more detailed and accurate picture of a student's ability to work in each van Hiele level, making it an excellent research tool, it is not suitable for use in the regular classroom in its present format.

This analysis not only gives us a clearer perspective about the evaluation method developed by Gutiérrez, Jaime and Fortuny, it also allows further insight into the van Hiele Theory. In particular, it provides further empirical evidence about the robust nature of the levels and about what it means to understand at a certain level.

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WHO PERSISTS WITH MATHEMATICS AT THE TERTIARY LEVEL: A NEW REALITY?[1]

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Abstract

Mathematics is widely regarded as a critical filter to educational and career opportunities. While in many countries more students now proceed to tertiary education, relatively few apparently elect to study advanced mathematics. In this paper we suggest that inaccurate participation and attrition data may inappropriately deter students from continuing with mathematics at the highest levels. By quantifying the drift away from mathematics more precisely we illustrate that mature age students and selected females are more likely to persist with mathematics. We include interview data to illustrate students' attitudes and their motivations for selecting mathematics.

Introduction

The study of mathematics is widely considered to be important and to serve as a critical filter to many educational and career opportunities, if not to the prosperity of the nation. "Mathematics and science have a fundamental contribution to make both to understanding the world and to changing the world, particularly in the context of change and economic adjustment" (Speedy, 1989, p. 14). Despite this it appears that interest in studying advanced mathematics and science has steadily declined.

> There are strong indications of increasing ... enrollment problems concerning mathematics and physics education as a rather international phenomenon. During the last years, reports of a significant decline in recruitment to higher education involving mathematics or physics have appeared in many parts of the world, including many countries in Europe, the USA, Australia, and Japan. (Jørgensen, 1997, p. 11)

The scope of this decline and the reasons for it are worth exploring.

Enrolments: new targets

In Australia, as in many other countries, various policies have been put in place to encourage young people to proceed to tertiary education. These changes seem to have been of particular benefit to two groups: females and older students. Females now substantially outnumber males at university (Australian Bureau of Statistics, 1996 and earlier years). Greater flexibility shown by tertiary institutions in their entry requirements has also been an advantage for mature age students, that is, students who

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have not necessarily completed the formal academic pre-requisites required generally, or for specific courses, and who are 21 or over on March 1 of the year in which entry is sought. Increases in enrolments have not been uniform across different subjects or courses, however.

Mathematics enrolments

Gender issues. Gender differences in performance on mathematical tasks and participation in optional mathematics courses and related activities have attracted much attention over the past two decades. A careful reading of the literature reveals that there is considerable overlap in the performance of males and females (see Leder, Forgasz & Solar, 1996 for a comprehensive review). More consistent gender differences continue to be reported, however, in participation patterns. Mathematics and related occupations have been identified as a male domain in many countries. Statistics in the USA and elsewhere reveal that women and minorities are under represented in the most advanced mathematics courses and in related professions.

Explanatory models. Various explanatory models have been proposed to account for the gender differences observed. They share many common features:

> the emphasis on the social environment, the influence of other significant people in that environment, students' reactions to the cultural and more immediate context in which learning takes place, the cultural and personal values placed on that learning and the inclusion of learner-related affective, as well as cognitive, variables. (Leder, 1992, p. 609)

Eccles et al. (1985) argued forcefully that an individual's choice of educational and career pathways selected is influenced not simply by reality but also by perceptions and interpretations of that reality. Thus students capable of continuing with mathematics, for example, but who believe that studying this subject is not appropriate for them are more likely to turn to areas they believe others consider more appropriate for them. Further credence to this hypothesis is given by descriptions of the subtle ways in which thoughts and behaviors interact:

> Attitudes involve what people *think* about, *feel* about, and how they would *like to behave* toward an attitude object. Behavior is not only determined by what people *would like* to do but also by what they think they should do, that is *social norms*, by what they have usually done, that is, *habits*, and the *expected consequences of behavior*. (Triandis, 1971, p. 14)

Intensive explorations of factors contributing to the reportedly high attrition rates among undergraduates in mathematics and the physical sciences are rare. Seymour and Hewitt's (1997) comprehensive examination of factors critical in the decisions of American undergraduates to switch from science, mathematics, and engineering courses into non-science disciplines is a welcome exception. An ethnographic approach, they argued, was most suitable for their three-year study since relatively little was known about the kinds of factors which might be involved. Nevertheless, their search of the relevant research literature did identify some critical elements, including differences between males and females in psychological alienation, self-esteem, career aspirations, and persistence. Seymour and Hewitt (1997) commented on the failure of many college and university departments to keep "enrollment, persistence, and attrition records" (p. 14). As discussed in the next section, obtaining an accurate long range overview of enrolment trends is problematic. Tracking such patterns for selected subjects is fraught with further difficulties.

Mathematics enrolments

Questionable data. Enrolment figures are collected annually from Australia's tertiary institutions. However, "much of the readily available statistical information aggregates the whole of science and therefore tends to mask trends in individual disciplines" (Australian Academy of Science, 1993, p. 4). Problems with more detailed coding required by government from institutions, for funding purposes for example, have also been identified:

> In general, the level III fields of study for mathematics were of limited value in considering either participation rates for students overall or relative rates for men and women. This stemmed from attempting to apply a vocational classification to a discipline in which most programs in which majors in mathematics can be completed are of a generalist nature.... Interpretation of the data was (further) clouded by the sudden appearance or disappearance of institutions in the fields over the study period. This in part stemmed from changes to course structures resulting in redefinition of the level III field. (Cobbin, 1995, p. 62)

Such incomplete tracking of participation rates not only affects the portrayal of the reality on which students base their decisions about appropriate areas of study, but also affects societal perceptions of reality.

A new approach. To supplement the broad national enrolment data available from government sources (Department of Employment, Education, Training and Youth Affairs [DEETYA], 1996), we sought enrolment information for 1996 and 1997 specific to mathematics by writing to the 38 Higher Education institutions across Australia. In particular we requested enrolment figures by student sex and age (under 21 or not in March of the year of initial enrolment). We also requested details about the DEETYA codes used to identify students likely to pursue a major in mathematics. Available codes included: mathematics, science-general, computer science, life sciences, and information systems. Twenty-three institutions supplied us with information. Of these, 15 replied with all the information in the requested format. Reasons for some of the omissions in other returns are clarified below.

Explanations for the (non) selection of the available DEETYA codes were informative. For example, one institution responded:

> We are unable to complete this section owing to the difficulty of obtaining details on majors at undergraduate level, until after the student has graduated.

Another wrote:

> All of our students' courses will be categorised as either "science-general" or "mathematics" depending on whether they have elected their major at the time of enrolment. Thus, first year students will usually be coded as "science-general" while later year students will gradually move to more specifically coded majors/degrees.

This latter approach was used by a number of our other respondents. In contrast, one institution coded all its mathematics students to the "life sciences", three indicated that they coded all students taking any mathematics subject as "mathematics", while others coded all such students as "science-general". These inconsistencies highlight the difficulties of obtaining accurate longitudinal, or even annual, data and suggest that the national data base underestimates the actual number of students studying and majoring in mathematics. When combined with global statements about the lack of appeal of mathematics to students, the picture presented may sway students' perceptions of mathematics as either appropriate or inappropriate for them. It is worth recalling the assertion by Triandis (1971) that "behavior is not only determined by what people *would like* to do but also by what they think they should do, that is *social norms*" (p. 14).

The data. The mathematics enrolment data, overall and for selected groups, are shown in Table 1. The data returned were from institutions with a total enrolment of 275,600 students in 1996, representing just over 40% of the 1996 national higher eduction enrolments. By comparing enrolment figures for 1996 and 1997 for this sample we were able to calculate continuation rates from first to second year and from second to third year.

The data in Table 1 indicate that:

- despite a greater number of females than males in Australia commencing first year university studies (DEETYA, 1996), more males than females studied at least one first year mathematics subject
- a higher proportion of males than females continued with a mathematics subject from first to second year, but
- a higher proportion of females than males continued with a mathematics subject from second to third year
- for both males and females, the attrition rate from first to second year, and from second to third year, was higher for younger students than for mature age students.

The continuation data for students from second to third year reveal a rarely reported high degree of persistence for a select group of females: those who studied university mathematics for at least two years. Also worth noting is the relatively strong persistence of mature age students in their mathematics studies. Interview extracts provide an informative context for these statistics.

Table 1. Selected enrolment and continuation data

Year 1	Total	Male			Female		
		All	<21 yrs	≥21 yrs	All	<21 yrs	≥21 yrs
1996	24725	14628	11189	3439	10097	7941	2156
1997	26422	15527	11896	3631	10895	8552	2343
% male (1996)		59.2	58.5	61.5			
% male (1997)		58.8	58.2	60.8			
Year 2	Total	Male			Female		
		All	<21 yrs	≥21 yrs	All	<21 yrs	≥21 yrs
1996	9998	6964	4686	2278	3034	1953	1081
1997	10004	6889	4746	2143	3115	2053	1062
% male (1996)		69.7	70.6	67.8			
% male (1997)		68.9	69.8	66.9			
Year 3	Total	Male			Female		
		All	<21 yrs	≥21 yrs	All	<21 yrs	≥21 yrs
1996	3268	1912	1058	854	1356	761	595
1997	3289	2041	1180	861	1248	704	544
% male (1996)		58.5	58.2	58.9			
% male (1997)		62.1	62.6	61.3			
% Continuing: 1st to 2nd year	40.5	47.1	42.4	62.3	30.9	25.8	49.3
% Continuing: 2nd to 3rd year	32.9	29.3	25.1	37.8	41.1	36.0	50.3

Student interviews. We interviewed mature age students as well as students who proceeded to university directly from school. The (representative) extracts below highlight the issues on which the differences between these groups were most marked:

reasons for studying mathematics, more positive attitudes to mathematics and to working hard, persistence, and a willingness to seek assistance.

The first set of extracts illustrates a fundamental difference, elicited in a number of interviews, between mature age students and school leavers for studying tertiary mathematics and with respect to their attitudes to this subject.

> I came back to study because my career choices were limited without it ...I spent most of that time [since leaving school] employed in the clothing industry and sometimes I worked for people and sometimes (I was) self employed ...and ended up just absolutely hating it and so had to sit down and look at my options and thought, well, further education was always one of my aims as a teenager. I was always going to go to university but I never had any real picture of what I was going to become which was the main problem when I was at school.... So basically it took me 10 years to realise that I just want to do mathematics. (Lara, mature age)

> Well with mathematics I have always had a very deep interest and quite like mathematics and when I finished my final year of school back in 1972 I had a very difficult year that year for a number of reasons and I failed the year and I always meant to go back to school and that never happened until this year and I had to sit quite a few tests before they would allow me in. The main reason I am doing it is because I have always had a deep liking for maths and I have always wanted to pursue it. (Dan, mature age)

> I guess that [mathematics] is probably the last option that I had because my mark [in grade 12] was not so flash and that was the last option there was for me with which I could get into university. So I guess I had to take that option. Mathematics is not a big overly difficult subject for me. I don't find it too hard. So I thought it would not be too bad to do... It is a compulsory subject... Even though I am interested in it, I am not too bad at it, but I don't think I would choose it. (Kip, school leaver)

> Honestly [I chose mathematics] because I don't like any of the other subjects offered .... And I thought mathematics would be a logical subject and all I have to know is how to use the formulas and I would be able to get along with it (Anne, school leaver)

Subtle differences between the more mature and younger students in attitudes to, and for, getting help are illustrated by the next two extracts

> Here (at university) it feels pretty good. I find probably just about every day, or may be three or four days a week, I will be knocking on somebody's door saying "I have got a problem with this" and I actually kind of like doing it because I like discussing the subject. Even if it is not that much of a problem it is good to discuss it. I find I don't fear it as much if I can talk about it (inaudible). So... they are usually helpful

> although sometimes you feel like you are bothering them (inaudible) and I probably am too. ...They are all pretty helpful. (Bill, mature age)
>
> I don't usually ask for help in that I prefer to get a textbook or something and get help there. So I don't usually see lecturers but other students are different. (Mark, mature age)
>
> At school the teachers are always there pushing you because they feel they have a responsibility.... whereas at university level there is help if you want it ... (but) you have to go and ask for help yourself. There is no help offered on a one-to-one basis. ...I have needed help in the past, yes, and it is sometimes difficult to find the relevant people you need to talk to but once you find them the help is generally available at tertiary level. ... It is easier to get help [in the school classroom] if you want it because you can get help then and there. (Sam, school leaver)
>
> I think in school it kind of unique because you have got one teacher and they have known you for a few years and so they are always willing to help and you can talk to them where at university it is more of a bureaucracy... It is, I find in mathematics that if you don't think you are getting enough help it is probably, or sometimes I don't think I am getting enough help, it is probably my fault more than theirs because they are always willing to be there. (John, school leaver)

Final comments

National enrolment data, from which reductions in the numbers of students engaged in high level mathematics studies are traditionally inferred, are not as precise as generally assumed and mask information about students who do persevere with mathematics. Our more painstaking efforts to gather accurate information identified difficulties associated with large data bases and highlighted selected groups who persist with their studies in this area. In interviews, these students - mature age students and females who have remained with mathematics for two years of undergraduate studies - seemed more convinced of the importance of mathematics to them, had a greater awareness of the intrinsic value of mathematics, were more task-oriented, willing to work hard and consistently, sought, and obtained help when needed. Societal and institutional preoccupation with those opting out of mathematics rather than with those who persist may subtly reinforce cultural perceptions of mathematics as a suitable area of study for only a small minority.

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RE-STARTING ALGEBRA IN HIGH SCHOOL:
the role of Systemic Thinking and of Representation Systems Command

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Abstract

In this paper we focus on restarting algebra in the first years of high school; in particular, we analyse connections between ability of thinking in systemic terms and algebraic modelling and discuss about the significant influence of students attitude and capability to make use of representation registers rich of operative potentialities on ability of algebraic modelling a situation; we finally suggest some didactic implications of our analysis on classroom activities.

Introduction

Mastering algebra is an aim of the high school (14-18 y.o. pupils), while a preliminary introduction to algebra is already made in the middle school (11-14 y.o. pupils), at least in some countries. In middle school, most teachers make a first introduction to some algebraic activities, mainly to the use of letters in expressions, but in non-homogeneous ways. This causes a problem at the beginning of high school, as students coming from different classes have different backgrounds, and especially different conceptions, on what algebra is. This algebraic background can vary from a minimum experience with literal calculus up to using algebraic formulas to describe some phenomena; nonetheless, more the procedural than the relational aspects involved are usually emphasised. The math teacher in the first year of high school has then the problem to restart algebra in such a way that each student can relocate what he already knows in a meaningful framework that makes him understand and appreciate the conceptual break with arithmetic.

Research problem

The way restarting algebra is developed at the beginning of high school is very important and may be the origin of some graduated adults' (teachers, engineers...) conceptions of algebra (Hudson et al.,1997), making it difficult to be used in professional contexts .

Mastering algebra implies both being able to model (Chevallard, 1989) a situation in algebraic terms, that is finding out and writing formulas representing the relations among the involved entities, and being able to manipulate expressions and relations by means of rules in order to achieve fixed aims.

It is well known that being able to manipulate expressions and relations does not imply being able to model a situation (Arzarello et al.,1994) and that some representations of a problem situation may be correct but not sufficient to result in a right solution of the problem, not even in arithmetic problem solving (Lemut & Mariotti, 1995; Dettori & Lemut, 1993).

Out of other authors, Rojano & Sutherland (Rojano & Sutherland, 1997) write that in algebraic approach to solving a problem it is necessary to be aware of and keep in mind the complete set of relationships present in the problem and to explicitly use a verifier of the attempts undertaken.

It is known that algebraic modelling of a problem situation implies a conversion[1] (Duval, 1994) from the text of the problem expressed in common language to an algebraic representation; but this conversion is not easy and is more difficult if it is a not-congruent conversion (in other words, if it is not an act of coding from one register to another (Duval, 1988)).

Finally, it must be taken into account what Yerushalmy (Yerushalmy, 1997) writes, mentioning Hall (Hall,1989), "I think, like Hall, that a major agenda of algebra teaching should be equipping learners with tools for mathematizing the perception of the situation context;..."; furthermore he underlines that the main research findings suggest that solvers of word problems in algebra devote a substantial portion of their work to re-representation of the problem at the situation level.

In this research context, it could be interesting to deepen what is at the basis of the problems re-representation processes activated by the students and which tools they should be equipped with, in order to highlight when and how activities of manipulating representations in different registers and of converting from a language into another lead or not to an algebraic modelling.

Research hypotheses

We shall develop the analysis of students' and teachers' problem solving processes according to the following hypotheses. We think that *algebraic modelling is based on the convergence of three major components: systemic thinking, mastery of a wide range of representation systems rich of rules for manipulating representations, capability of converting representations from one representation* system to *another* These three components can have, from time to time, different level of mutual influence and coordination.

The capability of systemic thinking is an educational aim which overcomes algebraic thinking and operating; it is a transdisciplinary aim. Licon Khisty (Licon Khisty, 1997), when discussing about how to reform mathematics education, describes *"systemic thinking"* as a general philosophy that suggests *thinking globally but acting locally*. Economists apply systemic thinking for designing and analysing economical scenarios.

Systemic thinking, in the context of learning/applying algebra, concerns at least two different aspects: modelling and manipulations control. As regards modelling, we think that systemic thinking intervenes in at least two cognitive activities: a) to identify the involved entities in a given situation, to decide for each entity if it is influenced by the others or influences them, to isolate each single entity in order to analyse how it is mutually linked to the others; b) to verify the perception, in a sequence of representations, of being or not in front of a representation that can be just converted into algebraic language. As regards manipulations control, we hypothesise, even if we do not discuss about it in this paper, that systemic thinking is also at the basis of controlling the sequence of results of successive manipulations in order to decide, from time to time, if the reached result is suitable to check the plausibility of a conjecture, to demonstrate a theorem, to highlight a property (Gallo, 1994).

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[1] "*Conversion* is the transformation of the representation of an object, a situation or a given information from one register to another.... Hence, conversion is an external transformation in respect of the first representation register." (Pavlopoulou, 1994), (translated from French)

In general, thinking in systemic terms, hence operating on relations in order to obtain new information, entails that one be equipped with representation "registers" (Duval, 1995) of different kind, each supporting a rich system of rules which allow to manipulate the represented relations. On the other side, algebraic competencies are usually fundamental for analysing in quantitative terms a set of relations expressed in some representation register.

In this paper, focusing on restarting algebra in the first years of high school, we analyse connections between ability of thinking in systemic terms and algebraic modelling and discuss about the significant influence of students attitude and capability to make use of representation registers rich of operative potentialities on ability of algebraic modelling a situation; we finally suggest some didactic implications of our analysis on the classroom activities.

Experimental setting

We planned some activities with about 40 forty 14-15 years old pupils of two first classes of a fine art lyceum in order to investigate on the above mentioned points. We verified previously that the type of school does not seem to affect students' attitude to make autonomous use of some representation systems. In none of the two classes pupils' algebraic backgrounds have been investigated; anyway, our aim was to restart algebra by means of problem solving activities, considering this context as the most adequate to develop modelling capabilities and to stimulate thinking in systemic terms. The official program of this kind of school and the number (4) of hours per week are only a bit different from those of other more demanding high schools.

Moreover, the same problems have been proposed to some math teachers in order to find out analogies or differences in experts' and novices' approaches to problem modelling.

The examined problem situations

Among the problem situations submitted to the students, let us show the following ones, which are significant as concerns connections among thinking in systemic terms, representing in various representation systems and algebraic modelling.

PROBLEM 1 :*"Mark's father is 38, Mark is 6. In how many years Mark's father age will be three times Mark's age?"*

Through this problem we wanted to investigate *whether when a problem is expressed in terms that are not-congruent in respect of possible formalisations in the representation register in which one is asked to solve it, it is possible to result in a solution, either algebraic or arithmetic, without thinking in systemic terms or making flexible use of some intermediate situation's representations.*

The teachers succeeded in modelling the situation by means of the equation "$38+X=3(6+X)$"; only some of them needed to produce some graphical representations (such as little segments) of the initial data and their modification year after year, and to operate on them by simulation of movements (physical comparison of segments).

The teachers, asked for an arithmetic approach to the problem, coped with some difficulties; furthermore, both pupils and adults were not able to easy understand the following resolution strategy: "*Today the difference between father's age and three times son's age is 20. Year after year, father's age goes on by 1, three times son's age goes on by 3, so that they get nearer by 2. Hence, the initial difference of 20 becomes 0 in 10 years.*". In fact, this arithmetic solution requires a systemic approach supported by some mental or external representation, that must be manipulated: globally thinking concerns the identification of a certain number of involved entities (father's age, son's age ,three times son's age, the difference today…, father's age after one year, son's age after one year, three times son's age after one year, the difference after one year…,.and so on); locally operating concerns deciding to calculate the initial difference, how much this difference decreases each year in relation to the difference year after year between father's age variation (which is 1) and three times son's age variation (which is 3), and in how many years the initial difference collapses.

As concerns what we wanted to verify, we feel that also he teachers who appeared not to resort to any visible representation manipulated in their mind the verbal text of the problem until they saw in their mind a new representation of the problem text, perceived as convertible into algebraic language in a congruent way.

PROBLEM 2 : "*An architect has to cover a 4 meters per 3 meters opening in a wall with a rectangular window having a wooden border of the same width on all the sides. For a reason we don't know, the window glass should have a 10 metres perimeter. How wide should be the wooden border?*"

Through this problem we wanted to verify *whether submitting to students problems whose text is not easily convertible into algebraic language, but is easily representable in a well known environment, within which they are used to write algebraic expressions, can be a good way to make students aware of the importance of resorting to rich representation systems as mediators towards algebraic modelling.*

Facing this problem, the teachers drew a rectangle of dimensions nearly proportional to the given ones and, inside it, another rectangle with the same centre; then they expressed a problem resolution in words (by mind or by writing), on the basis of the made representation and not of the text anymore; finally, they converted the resolution into algebraic terms by means of the equation "2(4-2X) + 2(3-2X)=10".

Most students made the same drawing, adding numeric data on it, but then they solved the problem by trial and error; by doing so, they were far from thinking both in relational and in algebraic terms.

The following resolution strategy proposed by a student is interesting to be examined; it must be taken into account that the teacher told his class that they were restarting algebra by means of problem solving activities. Besides a drawing similar to the above described ones, he wrote "*2p=(3+3+4+4)M=14M; 2pglass=10M; 10=(4-2x)+(3-2x)*2*". In this case, regardless to the standard error in the final algebraic expression, jumping to converting the situation into algebraic terms was been mediated by moving from the representation system of geometrical drawings to that of formulas aiming to highlight some properties of the drawings themselves;

in this moving, the student still made use of terms that were not at all algebraic terms: in fact, the used letters M and p and the string pglass are neither variables nor unknowns but only labels.

As concerns what we wanted to verify, we have to take notice that the answer to our hypothesis may be negative if students resort to representing outside of a deliberate systemic approach to the problem situation; in this case, for example, we feel that resorting to geometric representation has been quite automatic. It is also interesting to underline the importance of flexibly moving from a representation system to another, being always aware of the fixed goal (that is solving the problem through algebraic methods); this is one of the systemic thinking aspects.

PROBLEM 3 : *"In a house there are three little brothers. One night, one of them awakes and finds some chocolates on the table; he eats one third of them and comes back to bed. Later on, another one awakes and does the same thing, that is he eats one third of the chocolates he finds on the table. Again, the third little brother awakes too and eats one third of the chocolates he finds on the table. At the end, on the table there are 8 chocolates left. How many chocolates were there on the table at the beginning?"*

Through this problem we want to investigate on *which level of complexity in systemic thinking one, either student or teacher, is able to handle (at the moment).* (we chose a problem expressed in such a way that a congruent transformation can be activated into algebraic language just from the beginning)

One mastering fractions and algebra is supposed to immediately convert the problem into algebraic language by means of the equation
"$X-\frac{1}{3}X - \frac{1}{3}(X-\frac{1}{3}X) - \frac{1}{3}(X-\frac{1}{3}X- \frac{1}{3}(X-\frac{1}{3}X))=8$"".
Actually, this is not always the case: teachers too developed solution strategies in which not all the involved entities are put in relation since the beginning. For instance, after writing "$X-\frac{1}{3}$ ", some of them went on as follows:

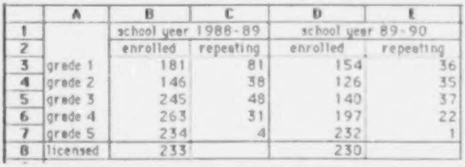
"$X-\frac{1}{3}X=\frac{2}{3}X$ - number of chocolates left after the first brother;
$\frac{2}{3}*X-\frac{1}{3}*\frac{2}{3}*X=\frac{2}{3}*X-\frac{2}{9}*X=\frac{4}{9}*X$ - number of choc. left after the second brother;
$\frac{4}{9}*X-\frac{1}{3}*\frac{4}{9}*X=\frac{4}{9}*X-\frac{4}{27}*X=\frac{8}{27}*X$ - number of choc. left after the third brother;
$\frac{8}{27}*X=8$ from which $X=27$".

Most students represented the situation by means of areograms, by congruent transformations, because of the presence of fractions in the text of the problem. The used representation suggested them some further manipulations according to the text of the problem (representation of the initial and successive quantities of chocolates), hence reinforcing the idea that the followed strategy was fruitful; but being unable to put in relation what already represented with the given numeric data 8 stopped the solution process in some cases. Both students succeeding in finding out the mentioned relation and those making use since the beginning of a representation including this relation (starting from a set of 8 objects divided into two sets each of 4 objects) solve the problem by means of the following arithmetic solution strategy:

"$8=\frac{2}{3}$; $8:2=4$; $4*3=12$; $12=\frac{2}{3}$; $12:2=6$; $6*3=18$; $18=\frac{2}{3}$; $18:2=9$; $9*3=27$; 27 is the initial number of chocolates".

As concerns what we wanted to investigate, we have to take notice some teachers seem to be able to master a level of complexity of systemic thinking comparable with this of the students who are able to put in relation the given 8 with the representation of the part left by the last brother, even if they resort to different representation systems. Teachers who operate step by step show both to think and to act essentially at local level.

PROBLEM 4: "*Let the following table be given, representing the number of enrolled students in the five classes of a high school during two consecutive school years. Write some relations to determine for every class how many students dropped out school between the two school years.*" (for simplicity, we assume there hasn't been any students interchange among different schools) - (from the project MaCoSa, by C.Dapueto)



	A	B	C	D	E
1		school year 1988-89		school year 89-90	
2		enrolled	repeating	enrolled	repeating
3	grade 1	181	81	154	36
4	grade 2	146	38	126	35
5	grade 3	245	48	140	37
6	grade 4	263	31	197	22
7	grade 5	234	4	232	1
8	licensed	233		230	

Fig. 1

Through this problem we wanted to investigate which *kind of behaviours could be activated, expecially after giving a systemic representation of the situation, when in the text of a problem not all the involved entities are mentioned.*

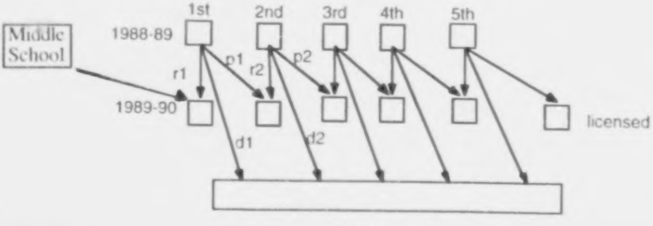


Fig. 2

All students tried to solve the problem by writing out formulas in the spreadsheet environment which they had previously worked in and whose richness, as concerns allowed manipulations, they had appreciated (it must be noted that this problem was submitted after working for two months in a spreadsheet environment, where students were asked to interpret and write some algebraic formulas). Only one out of 40 wrote the formula "=D4-E4" in order to express passed students (not-mentioned entity in the problem text, neither in the verbal part nor in the table included), but then he was not able to go on.

Once a representation of the problem situation by means of a graph was given to students (Fig.2), starting from this representation and its rules (the value of any node is the sum of the values of all arrows from it or to it) and basing on data representation in the table, most of them developed a problem solution in algebraic terms by means of the following relations *"D4=P+E4; B3=E3+A+P"* (where P: passed and A: drop outs); this model of the problem situation can be considered algebraic especially because formulas are not expressed in procedural terms, such as "P=D4-E4; A=B3-E3-P". Then students succeeded in algebraic modelling the problem situation without difficulty; at this point, they were only asked to convert into algebraic language what was expressed in graphs language in such a way that was congruent with it.

The teachers preferred not to use the offered graph representation after deriving from it the importance to consider "passed students". One of them, after producing the solution *"181-8; passed 1°-2°; 181-91; 90 36 repeating; 90-36 54"*, guided her students to make use of the graph representation in a systemic way.

As concerns what we wanted to investigate, we point out that nobody approached the problem in a systemic way before having in his hand the graph representation; then, even if with some initial reluctance, the teachers felt the importance of teaching to look at a problematic situation by thinking globally and acting locally; the students used without difficulty the graph as a basis for converting some parts of it into algebraic language.

Finally we submitted also the following PROBLEM 5: *"Five friends have debts/credits each other. Anton has to give Daniel 40$, Daniel has to give Eric 80$, Charles has to give money to both Daniel and Eric but he doesn't remember how much he has to give to each of them, Bernard has to give Daniel some money but he doesn't remember how much, Charles has a total debt of 30$, Daniel knows his debts are equal to his credits, Eric knows he should receive 100$ at all. Which is the debts/credits situation of the five friends?"* for showing the importance of systemic thinking and representing when the network of the relations is too complex.

Conclusions

In the previous paragraph, through the analysis of some problem solution processes, we have highlighted the connections between mastering algebra and being able to think in systemic terms; we have also shown how these two aims, a disciplinary and a transdisciplinary one, are strictly connected with making conscious use of representation systems of different kind, each supporting a rich set of rules which allow to manipulate the represented relations.

In particular, we want to further underline the following items: a) thinking in relational terms is not an exclusive feature of algebra, so that it can be developed before or besides algebra (problems 1 and 3); b) resorting to representation systems without aiming it to find out the relations among the entities involved in a problem situation can be of not use at all towards algebraic modelling (problems 2 and 3); c) moving from a representation system to another, being always aware of the fixed goal, that is solving the problem through algebraic methods, is crucial (problem 2); d) when the algebraic model of a situation is too complex, even if the text of the problem is expressed in such a way that is congruent with algebraic language, also math teachers tend to represent the situation step by step using a mixed language, which is halfway algebraic and halfway verbal (problem 3); e) when the involved entities are not all mentioned in the text of a problem or the network of their relations is too complex, algebraic modelling requires the support of some intermediate representation system (problems 4 and 5); f) being used to different software environments may develop attitude to think that the represented objects may be manipulated and anyway must be manipulated according to the rules of the environment itself (problem 4).

Concluding, we think that restarting algebra in the first years of high school could be more effective in a problem solving context, paying attention to what kind of problems are meaningful and focusing first of all on modelling the problem situation, that is on finding out and making explicit the relations among the involved entities; this approach doesn't entail graduating the situations on the basis of the number of equations or inequalities arising from the modelling process and of the complexity of manipulations to be made on them. Furthermore, modelling and manipulating processes are better distinct; students attention can be focused more on structural than on numerical aspects of problems resolution; learning manipulation rules can be effectively motivated by being the tools which allow students to make explicit the solution already implicitly encapsulated in the model of the problem, beyond finding out new properties of the involved entities or new meanings of the involved algebraic expressions.

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IN PURSUIT OF PRACTICAL WISDOM IN MATHEMATICS EDUCATION RESEARCH

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ABSTRACT

Currently there is a serious and growing lack of communication among researchers in mathematics education as well as between researchers and education practitioners. In this paper I draw upon my experience as editor of the Journal for Research in Mathematics Education to challenge the mathematics education community to begin to consider how we might establish real lines of communication among researchers and between researchers and practitioners. Of particular, interest to me is the development of "practical wisdom" about the contemporary problems confronting us.

During my term (1992-1996) as editor of the *Journal for Research in Mathematics Education (JRME)*, there were numerous instances of an apparent failure of authors to communicate appropriately with their intended readers. Many of these instances were due in large part to the lack of shared principles among researchers regarding such fundamental matters as what counts as research, the role of justification, what counts as evidence, and the place of background assumptions and beliefs in the research process. An even more serious lack of communication exists between researchers and teachers, teacher educators, and other educational practitioners. In this paper I make some suggestions about how to increase the level of communication.

Lack of Communication Among Researchers

There is reason to believe that the communication gap that exists within the research community will continue to widen for some time to come. In my first editorial, I observed:

> Many educational researchers have come to embrace methodologies developed in such disparate research disciplines as anthropology, psychology, history, philosophy, and sociology, as well as the various "natural" sciences.
>
> This poses a dilemma for reviewers (as well as for editors) because researchers and academics within any discipline set standards for scholarly discourse that often are not functional outside that discipline. Academic disciplines can be distinguished from each other in some fundamental ways. In particular, they differ with regard to the nature of the questions asked, the manner in which questions are formulated, the process by which the content of the discipline is defined, the principles of discovery and verification allowed for creating new knowledge within the discipline, and the criteria used to judge the quality of research within the discipline.
>
> When we decide to borrow from a discipline other than the one in which we were trained we must take care to give ample attention to the perspectives and assumptions underlying that particular discipline. Unless researchers who use methods outside their own research tradition have developed a good sense for these kinds of issues, it is likely that their research (and their assessment of research reports) will not be well-informed. (Lester, 1993, p. 2)

And, the dilemma is not restricted to reviewers and editors! The scope of methodologies, epistemological stances, and ideological perspectives found in the journal's articles have made it difficult for even the most capable and persistent readers to understand or fully appreciate all that appears in a given issue. Small wonder that a communication gap has developed.

Of particular interest to me are the difference positions that exist among researchers with respect to the role of justification in research—positions that have become apparent to me as I read reviewers' comments on manuscripts (and authors' responses to those comments).

Why Is Justification So Important?

Researchers concern themselves at various stages in their work with justification of at least three kinds: (1) justification of the importance and significance of their research agenda or specific research problems, (2) justification of the methods and methodologies they use to study their research problems, and (3) justification of the claims they make and the conclusions they draw.

Although some might think there is no controversy about the importance of these three kinds of justification, consider the fact that within the past 10 years or so, most, if not all, of the virtues researchers have traditionally looked for in educational research have been called into question. For example, some educational theorists, notably Lather (1994), argue that even so fundamental a standard as clarity of discourse might be regarded as problematic.

To further establish a case for the need to discuss justification, consider the editorial in a recent issue of *Educational Researcher* (Donmoyer, 1996). In this article the Features Editor pointed out that peer reviewers' recommendations often conflict and their advice to the author is sometimes contradictory. He went on to note that:

> Diverse recommendations from reviewers reflect, in part at least, the fact that ours is a field characterized by paradigm proliferation and, consequently, the sort of field in which there is little consensus about what research and scholarship are and what research reporting and scholarly discourse should look like. (p. 19)

The emergence within the past 15 years of "narrative research" illustrates Donmoyer's point. In her excellent review of this relatively new form of academic inquiry, Casey (1995) included a wide variety of practices (e.g., analysis of autobiographies and biographies, life writing, personal accounts, narrative interviews, ethnohistory, ethnobiographies, and popular memory). To this list, we might add "fictional narratives" (Eisner, 1993). Although no narrative research reports were submitted for publication in *JRME* during my term as editor, I would not be surprised to learn that current editor, Judy Sowder, has received, or soon will receive, such a manuscript. How will reviewers of a narrative report judge whether the author has provided, for example, adequate justification for her or his claims?

Finally, a common shortcoming of manuscripts submitted to *JRME* is the failure of authors to demonstrate how their conclusions relate to their conceptual or theoretical framework or to some set of underlying assumptions, and to provide an argument, situated within a set of background assumptions and beliefs, linking the claims they are making to the evidence they have presented. A considerable number of authors of relatively traditional research reports seem not to appreciate the importance of justifying their conclusions. For these researchers it is apparent that they believe "the data speak for themselves."

The Justification of Claims

A discussion of the justification of claims based on evidence is central to all the sciences—be they natural, behavioral, or social. Whether or not to believe, or even to take seriously, a scientific claim depends on the quality and nature of the evidence in its favor as well as on the nature of the case the researcher makes for the claim. The choice between conflicting theories—for example, between Newtonian and Einsteinian physics—cannot be made without examining the evidence put forward for each. If no evidence can in principle be presented that favors one theory over another, or if what counts as evidence is completely unclear, then the theory is considered unscientific. For example, one reason that scientists reject astrology is that many of its claims are so vague that it is unclear what would count as evidence that they are true or false.

But in this paper I am not concerned with what counts as evidence. Rather, I am interested in how scientific researchers go about justifying the claims (conclusions, hypotheses, implications, etc.) they make once they have identified the evidence. The relation between observational data and claims is determined by chains of reasoning situated in the set of background assumptions and beliefs operating in the context in which the data are being assessed (sifted through, categorized, classified, analyzed, etc.). Furthermore, however we end up characterizing observational data, they are what serve as evidence for the claims that result from our research efforts. Data—even as represented in descriptions of observations and experimental results—do not on their own, however, indicate the claims for which they serve as evidence. That is, claims are not "indicated" by data at all; conclusions are drawn or reached by researchers by means of chains of reasoning. Moreover, these chains contain premises. The empirical data merely constitute one set of premises, but there are others as well—there are always premises based on background assumptions and beliefs—that provide a link between the data and the conclusions.

Background assumptions or beliefs, then, are expressed in statements that are required in order to demonstrate the evidential import of a set of data to a claim. As such, they both facilitate and constrain reasoning from one category of phenomena to another. Relativizing evidential import to background assumptions and beliefs thus involves accepting this relation as involving substantive assumptions. Evidential relations are not autonomous or eternal truths, but are necessarily constituted in the context in which data are being assessed. Put another way, observational data do not in themselves point anywhere; they do not contain labels indicating what they are evidence for. Instead, they depend upon the context in which they are being considered and the connection made between the evidence and the claim.

But what about the rational process of moving from evidence to claims? Consider the following situation. Suppose a student were to stop by my office to tell me that she will not be in class today because she has a fever (confirmed by a thermometer reading of 40°C). I sympathize with her and suggest that she must have the flu and should go to the Student Health Center on campus. Is the assumed fact that she has a fever evidence for my claim that she has the flu? It depends! Specifically, whether this information constitutes evidence depends upon what explains why I regard it as evidence; that is, the rational process used to link evidence to claim. Suppose I offer one of the following two explanations:

A. I consider the student's fever as evidence of the flu based on my belief that a fever is a common symptom of the flu.

B. I base my claim on the fact that a crystal ball reader told me that a student with a fever would contact me to let me know she would not be in class and that this student would indeed have a case of the flu.

In both cases what is taken as evidence is the same: the student has a fever. But, what explains why it is evidence differs: In explanation A I believe that a fever is a symptom of the flu, and in explanation B, presuming that crystal ball readers are a reliable source of information on personal health matters, I believe what the reader told me. Furthermore, the alleged relation between the evidence and the claim differs in the two cases: in A I believe there to be a relation between having the flu and having a fever, and in B I believe there to be a relation between the crystal ball reader's predictions and what eventually happens.

The foregoing example serves to point out that rationality is not the infallible road to truth or away from error that it is often claimed to be. For example, Kuhn (1970) has suggested that an Aristotelian and a Galilean physicist looking at a swinging stone (pendulum) notice different things: the Aristotelian sees a body undergoing constrained fall, whereas the Galilean sees oscillatory motion—that is, a pendulum. Both the Aristotelian and the Galilean are being rational when they defend their respective accounts of a swinging stone. What, then, explains how rationality can serve to establish evidence for different claims? The Aristotelian and Galilean physicists attend to different aspects of the phenomenon because they hold different background assumptions and beliefs, which causes the stone's evidential relevance to be determined differently.

Once it is accepted that the evidential relation is always determined by background assumptions and beliefs, then it is not difficult to imagine that there could be a neutral description of a given set of circumstances, that is, one agreed to by both parties to a dispute, and no agreement on the claims for which it is taken as evidence. It is also easy to see that both parties could be perfectly rational. It is rational to take some set of circumstances as evidence for a claim in light of the background assumptions and beliefs one accepts. It would be irrational to assess evidential relations in a manner inconsistent with such background assumptions and beliefs and non-rational to accept or reject claims with no regard for evidence.

Tying this discussion back to the original concern about the lack of communication among researchers, let me note that background assumptions become problematic when, for example, they are based on metaphors and not directly subject to empirical inquiry. If the difference in assumptions is so slight that it could be resolved by simply pointing out an overlooked feature, there is no problem. But, if the difference is at the metaphoric level, there often is a definite failure to communicate.

In conclusion, I offer the simple suggestion that researchers devote more attention in their research reports to discussing the background assumptions and beliefs that are driving their choice of evidence and subsequent claims. Furthermore, reviewers (and readers in general) of these research reports should realize that observational data do not speak for themselves but are determined by the context in which they are being considered and the connection made between the evidence and the claims made.

Lack of Communication Between Researchers and Practitioners

> The *Journal for Research in Mathematics Education*, an official journal of the National Council of Teachers of Mathematics, is devoted to the interests of teachers of mathematics and mathematics education at all levels—preschool through adult. (inside front cover of every issue)

I am distressed to say that teachers and teacher educators do not pay much attention to the research so carefully and thoughtfully reported in the journal. Indeed, I would not be surprised to find out that the current level of communication between researchers and practitioners is as low as at any time in the history of mathematics education research. Among the many explanations for the failure of our research to resonate with teachers that have been proposed, one that has not been given adequate attention by math educators has to do with the fact that researchers and teachers have accepted different ways to frame their discourse about what they know and believe about mathematics teaching and learning. By and large, teachers communicate their ideas through, what Schwandt (1995) calls, "the lens of dialogic, communicative rationalism" (p. 1). By contrast, researchers typically communicate their ideas in terms of (monologic) scientific rationalism. Let me briefly discuss each approach.

Scientific Rationalism

For Schwandt, a research methodologist and educational evaluation specialist, scientific rationalism is a style of inquiry shaped by six principles:

1. True knowledge begins in doubt and distrust.
2. Engaging in this process of methodical doubting is a monological activity.

3. Proper knowledge is found by following rules and method (rules permit the systematic extension of knowledge and ensure that nothing will be admitted as knowledge unless it satisfies the requirements of specified rules).
4. Scientifically respectable knowledge depends upon justification, or proof.
5. Knowledge is a possession and an individual knower is in an ownership relation to that knowledge.
6. In justifying claims to knowledge there can be no appeal other than to reason. (Schwandt, 1995, pp. 1-2)

Of special concern for scientific rationalists is the nature of the claims that are made and how these claims should be justified. Furthermore, all the ways deemed acceptable for justifying a claim are regarded uncertain or unreliable in one way or another. Schwandt identifies four basic methods of justification: (a) argument by example to arrive at some sort of generalization, (b) argument by analogy (The argument goes something like this: because phenomenon A is like phenomenon B in certain ways, they are also alike in another specific way of interest to the researcher.), (c) argument from authority (the use of existing literature to support a position or help make a case); and (d) arguments from statistical inference. (Examples of the use of each of these methods of justification would be easy to identify in many issues of the journal.) Finally, any of these methods of justification is readily subject to the error of reaching a conclusion with insufficient evidence or to the error of overlooking alternative explanations.

Schwandt suggests that as useful as scientific rationalism might be for research purposes, it is not the only way to think about the important concerns surrounding making and justifying (evaluation) claims.

Dialogic Rationalism

Schwandt (1995) insists that dialogic rationalism opposes scientific rationalism in three fundamental ways. First, rather than regarding the social world as "simply out there waiting to be discovered" (p. 6), the dialogic rationalist insists that the world can only be studied from a position of involvement within it. Second, "knowledge of [the] world is practical-moral knowledge and does not depend upon justification or proof for its practical efficacy" (p. 7). Third, "we are not in an 'ownership' relation to such knowledge, but we embody it as part of who and what we are" (p. 7). Thus, dialogic rationalism provides a different way to consider what it means to know. "Instead of simple observational claims about objects, knowing other people is offered as a paradigm for knowledge" (p. 7). Schwandt suggests that to Ryle's (1949) two kinds of knowledge—knowing that and knowing how—we should add a third type: "knowing from." This type is characterized as knowledge "one has from within a situation, a group, a social institution, or society" (p. 7).

To accept dialogic rationalism involves accepting that reason is communicative: "It is concerned with the construction and maintenance of conversational reality in terms of which people influence each other not just in their ideas but in their being" (Schwandt, 1995, p. 7). It aims to actually move people to action, in addition to giving them good ideas.

Dialogic rationalism, then, has something to say to mathematics educators about how we make and justify claims in our research. In particular, dialogic rationalism attempts to avoid treating students and teachers as objects of thought in order to make claims about them that will guide future deliberative actions. Instead, it aims to include teachers (and students?) in dialogical conversations in order to generate practical knowledge in specific situations. Thus, claims are made only after the various perspectives (or world views, background assumptions and beliefs, etc.) of all those engaged in the dialogue have been openly considered and negotiated. Schwandt believes that is this process of open negotiation of claims (and of what is regarded as evidence) among all participants in the discourse that leads ultimately to practical wisdom.

In Pursuit of Practical Wisdom

I am not arguing that we should abandon concern for careful argument and evidence in favor of some sort of fiery, political rhetoric devoid of reason. What I am promoting is a renewal of a sense of purpose for our research activity that seems to be disappearing: namely, a concern for making real, positive, lasting changes in what goes on in classrooms. This is essentially what I mean by the pursuit of practical wisdom. To do this we might find it helpful to begin to move away from our preoccupation as researchers with the pursuit of "knowledge" (i.e., collections of bits and pieces of generally agreed upon information) and toward the goal of actually moving people (teachers, teacher educators, school administrators, policy makers, etc.) to action, in addition to giving them good theories.

Philosopher Richard Rorty (1979) offers another, related way to think about the dialogues that need to take place (a) within the research community, (b) within the community of practitioners, and (c) between these two groups. Specifically, he embraces postmodern philosophy as a voice in the ongoing conversation about what it means to be human. Within this conversation he distinguishes between analytical philosophy and hermeneutic philosophy. In an analytic endeavor, the participants are seeking to extend a scientific rationalistic account of some phenomenon and may indeed conceive of themselves as producing eternal knowledge. In hermeneutic activity on the other hand, the conversants seek only to extend the conversation in ways that enable them to better cope with some phenomenon in the present—not to establish an eternal body of knowledge. This form of discourse is essential to the development of practical wisdom; that is, ethically informed, reasoned conversation between researchers and practitioners (and among researchers) about issues that are fundamental to teaching and learning mathematics in contemporary society.

Rorty paves the way for Gallagher's (1992) assertion that true education, or edification, whether of school children, practitioners, or researchers, is a hermeneutical act always involving an interchange between at least two interpretations of some phenomenon. Such an assertion clearly brings us full circle to the roles our assumptions and beliefs play in the claims we make and in determining who will be moved by our claims. Returning to whence we began, we can now see one possible way in which to more fruitfully conceive of our research agenda and its ultimate aim. Practical wisdom is enhanced when the conversation about its focus is expanded in a rich and complete manner paying attention to the multiple meanings and interpretations (including beliefs and assumptions) brought to the discussion by each participant. That a conversation is the means to enhanced practical wisdom should not surprise us since ours is a practical wisdom concerning human beings. After all, why shouldn't getting to know our subject matter be more like getting to know a person than coming to formulate a premise?

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MORE GAMES FOR INTEGERS

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ABSTRACT

This is the final research report of Games for integers: conceptual or semantic fields *[Souza et al. 1995]. In that paper we reported on the first applications of three games: the butterflies, the gains-and-losses, and the snails games. The last one turned out to be somewhat awkward for use in the classroom and was substituted by two others: the macaws game and the game of bets. In designing the games we have been guided by Vergnaud's theory of Conceptual Fields and sought a didactic strategy that could lead the student to provide his/her own explanation for the sign rule. The objective of this paper is to report on a systematic application of these four games in real classrooms and discuss their didactical and pedagogical value.*

1. Introduction

Negative numbers have scarcely been dealt with in recent literature on mathematics education. Among fifty-six research reports presented in PME-18, only one explicitly concerns integers [Lytle, 1994]. In PME-19, there were two out of seventy-seven [Borba, 1995; Souza et al., 1995], one out of 160 in PME-20, [Bruno & Martinon, 1996]; and none in PME-21. Negative numbers have seldom been dealt with as a topic in algebra [VIENNOT, L. 1980]. In algebraic treatments a single letter represents both, a number and a predicative sign incorporated in a single proceptual unit, the *unknown*. Only operative signs are written down. Looking for such a signed-number as the solution of an equation, presupposes a certain familiarity with this sort of object. Our efforts in developing this research are based on the belief that familiarity with negative numbers persist as a necessary step towards algebra.

In Souza et al [1995], "we thought of anticipating the solution to the sign rule problem as *theorems in action*. Our idea was that roles should be exchanged: the teacher should be the one to ask and the student the one to answer why *minus times minus makes plus*. The didactical strategy should lead the student to provide his/her own explanation to facts that s/he should consider as evident" [Souza et al. 1995, p. 233]. In order to reach this objective we designed three games to solve in action four problems **P1**. How to take the bigger from the smaller? $(3-5=...)$; **P2**. How to subtract a negative? $(-(-3)=...)$; **P3.** What does "*minus ... times something*" *mean*? $((-3)\times ...?)$, **P4.** Why does minus times minus equal plus? $(-2)(-3)=...$ At that occasion we were guided by two pedagogical beliefs rooted in two questions of Vergnaud [1990]. Belief 1: By engaging the student in *games* where the use of theorems in action leads to better playing strategies, we could make theorems become theorems in action. Belief 2: By introducing adequate worksheet activities based on the game, after it has been finished, we could make theorems in action become theorems.

One reported result of the two 1995 pilot experiences was confirmed by research carried out during 1996 and 1997: the will to win does not lead students to use composition of additive operators as a theorem in action. They adhere to the first strategy

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[1] Partial support from CNPq and CAPES

they find. They need assurance that a more abstract strategy exists, a demand to reach it and, in some cases, hints. Therefore the first belief is not confirmed.

The pilot experiences reported in Souza et al [1995] referred mostly to the additive structure of integers. Results obtained from experiences with the multiplication structure were meager and could not confirm our hope that the games would lead the students *to provide their own explanation of the sing rule in real classroom situations*. This paper reports how this outcome could be obtained. It introduces two new games to substitute for one of the games suggested by Souza et al [1995]. Since the study's condition were quite adverse, we expect that such an outcome can be obtained more easily in normal classroom situations.

2. The multiplication structure of integers.

Historical difficulties with the conception of negative numbers are considered in Glaeser [1981]. Literature about integers provides models for the additive structure of integers [Thompson & Dreyfus, 1988] but is rather insufficient when considering the multiplication structure. Freudenthal proposes to insist on the necessity of permanence of distributive and commutative laws [Freudenthal 1973, p. 280] and to use the *geometric-algebraical permanence principle* [Freudenthal, 1983 p. 444]. After introducing the additive structure of integers through pairs of natural numbers, Dienes acknowledges the necessity of considering multiplication. He writes:

> "Before we can truly speak of spaces, we should invent at least one new operation: a mathematician would not call *space* what we have built up to now, and this because we have not invented any kind of multiplication; we have only invented multiplications and subtractions. In other words, if we start with $({}^{+}2, {}^{-}1)$, for instance, we should be able to double it to get $({}^{+}4, {}^{-}2)$ or triple it to get $({}^{+}6, {}^{-}3)$" [Dienes, 1972, p. 103, our translation].

However, if we look for the decisive point where the multiplication by a negative appears for the first time, we find:

> *Implicitly it is easy to see that* multiplying by negative numbers would reduce to double, triple, quadruple, etc. but this for each color separately, triangles and squares being interchanged [Dienes, 1972, p. 104, our translation, emphasis added].

It seems that this author does not give to the multiplication structure the same concrete development that he gives to the additive structure of integers. The question why minus times minus makes plus remains unanswered.

From the whole discussion we got the idea that no positive explanation ultimately exists and that some degree of arbitrariness is necessary. But, precisely where? We contend that arbitrary rule teaching is exactly the route to failure. This attempt leads the student to ask *why does minus times minus equal plus?* Then it is already too late; he has learned the solution without knowing the problem and is "fed up" with rules.

3. Four games for integers

Here is an attempt to solve problems P1 to P4. For a more complete description of the first two games we refer to Baldino [1996] and Souza et al [1995].

The butterflies game is an additive state-operator game intended to solve P1 (see figure 1).

The butterflies game

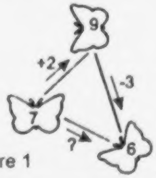


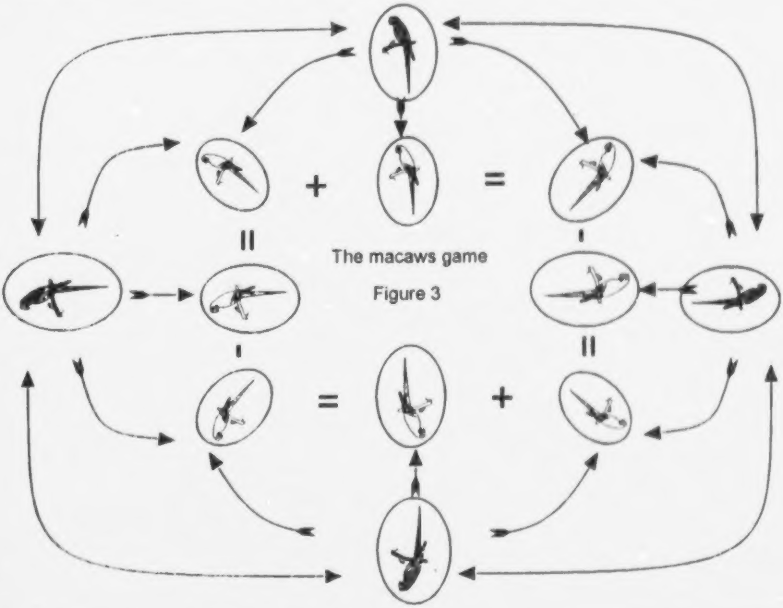
Figure 1

The game of bets

A + B = C

D - E = F

Figure 2



The macaws game

Figure 3

Creating worksheets from the macaws game

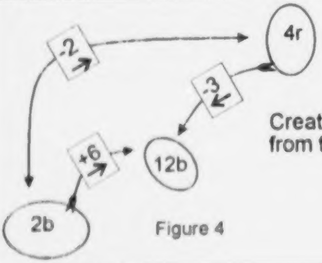

-2
4r
-3
+6
12b
2b

Figure 4

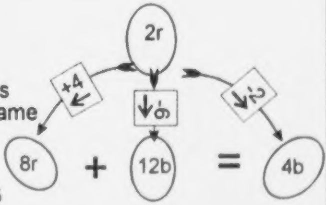

2r
+4
-6
-2
8r
+
12b
=
4b

Figure 5

The game of gains and losses is a real estate sales game with blue bills representing money and "red money" representing debts. It solves P2.

The macaws game is a multiplication state-operator game designed to solve P3 and P4. It consists of beads, cards and a board (see figure 3). The cards are stamped with an arrow and numbers from zero to three, preceded by + and – signs (see figures 4 and 5). The beads are blue (b) and red (r). The players must place the cards on the trajectories connecting the macaws, matching the card's arrow with the trajectory's arrow. The player who puts a card must also fill the macaws connected by the card with beads according to the following rule: If the card's arrow points from macaw M1 to macaw M2, the number of beads in M1 times the card's number must be equal to the number of beads in M2; the color of the beads in M1 must be equal to the color of the beads in M2 if the card has a plus sign and must be *different if the card has a minus sign*. Cards that result in fractional numbers of beads must not be played.

The game of bets is an auxiliary game for the macaws game, intended to be played before it. This game is designed to ease the difficulty with addition of multiplication operators, such as "four times, minus six times, is *minus two times*" (see figure 5). It consists of a board (see figure 2), instruction cards and fake money, blue representing gains and red representing losses, as in the game of gains and losses. The player follows the instruction of a randomly picked card. Here is a typical instruction: "Put your bet in A. Your first partner puts 10B (or 10R) in B. Your second partner puts the necessary amount to make the addition (or subtraction) exact in C. You collect what is put in C, your first partner collects what is put in B and your second partner collects what is put in A." The other cards contain similar instructions with the letters permuted, so that actually, in the long run, nobody wins.

4. The classroom study

Duration: The four games were used as a regular teaching device in three regular classes of a public school, 5th, 6th, and 7th grades, of lower-middle class students, aged 10-13 years. The city was Rio Claro, SP, Brazil, where a State University (UNESP) maintains a graduate program in Mathematics Education. One of us was the teacher in charge of these classes. Mathematics classes met three days a week during six 45-minutes periods. We organized the forty students into groups of four, with a plenary meeting once a week. Grades for participation were given out every day. A pilot study with the butterflies game and the game of gains and losses carried out in 1996 had resulted in a teaching failure. After each game had been tried for a couple of weeks, we introduced worksheet activities. It turned out that the children had to resort to the cards in order to solve the activities. Most of them became uninterested. Our evaluation was that we had introduced the worksheets too soon. Therefore, in 1997 we decided that the games would be played for their own sake. We would let the *vertigo of the rule* take the students [Baudrillard. 1979]. We decided to introduce a new game only after the students produced some evidence that they were tired of the old one. We decided not to worry about the syllabus. Only at the end of the experience, if we had time, would we introduce worksheets and go on to mathematical notation.

The following chart came out:

Study duration (in days)			
Game \ Duration	play	activities	total
Butterflies	colors: 6 signs: 7	3 8	9 15
Gains and losses	8	1	9
Game of bets	9	6	15
Macaws game	8	7	15
Total	38	25	63 days

The population: The conditions for the study were extremely adverse. In 1997, a government measure decreed that students could not be given failing grades. Unless the student drops out of school, credit was to be automatic. A final high school exam will be instituted in the near future. This measure only hides the symptoms of a deteriorating situation that has been going on for quite some time. In the schools of this region, it is usual to find children who cannot write their names being passed up to the third grade and adolescents finishing high school without being able to extract any meaning from what they read. Students are generally uninterested. Some schools are ruled by gangs. Destruction and stealing of school material is frequent. It is usual to dismiss classes due to teachers' absence. Students gladly go home earlier. Any motive is good enough to dismiss classes: holidays during the week, parties, parades, teachers' meetings, strikes... As a result, about 30 out of 60 school days were lost and we had to extend the scheduled duration of our study from one to two semesters. Traditionally, fulfillment of syllabuses are only questioned when the student gets a failing grade and the parents file a complaint against the teacher in the school district. Then the general verdict is the following: "If your have not covered the whole syllabus, you cannot assign a failing grade to the student".

As a consequence, the students that we found at the beginning of our study were completely unmotivated. They were used to copy unproductive tasks from the blackboard under an authoritarian look from the teacher. The physics teacher of a later grade told us that she gave up assigning problems to the students because they started up calculations without reading them and kept asking her: *Is this the way to do this one?* We could identify three large groups in the classroom. A few students (group 1) take the school seriously, as though in a stereotyped way. Other students (group 2) apparently take pleasure in challenging the teacher's authority, in spite of knowing that she is impotent to punish them. The majority (group 3) does not engage in either of these two strategies. It seems that the status obtained from being at school is sufficient for them. They love taking home high grades at the end of the year, in spite of knowing that they would get them anyhow. It seems that keeping the game of authority and pass/fail going is the major concern of all.

Worksheets: After the four games had been played for 38 days, the worksheet activities followed for another 25 days. Worksheets schematically reproduce parts of the game's board, such as in figures 4 and 5. For the macaws game, the task consists of reconstructing two of the following representations, given the third one: the *diagram*, the calculation *according to the cards, and* the calculation *according to the beads*. The students recognized this last modality as their natural way of playing and used it to check the others. Finally written situations that did not have a direct counterpart in the game were introduced.

5. Results

A research report should not be a bulletin of victory in teaching. Quality of research should not be confused with quality of teaching. Nonetheless, at least considering the extremely adverse teaching situation, our report is positive. We faced two challenges: a pedagogical and a didactical one. The pedagogical challenge was to keep children in school working on some productive cultural activity. The didactical challenge was to teach negative numbers so as to arrive at the activities proposed in the textbooks. We think that both challenges were met.

Sample of a worksheet

The diagram:

$$\begin{tikzcd}
 & & (-2) \arrow[dll, bend right] \arrow[d] \arrow[drr, bend left] & & \\
(+4) \arrow[d] & + & (-6) \arrow[d] & = & (-2) \arrow[d] \\
(-8) & + & (+12) & = & (+4)
\end{tikzcd}$$

Calculation from the cards: $(-2)\times((+4)+(-6)) = (-2)\times(-2)=(+4)$

Calculation from the beads: $(-2)\times((+4)+(-6)) = (-2)\times(+4)+(-2)\times(-6)= (-8)+(+12)=(+4)$

We initially explained the children how we intended to work and how grades would be assigned: part from a written final and part from engagement in the games during classes. This introduction was sufficient to produce a deep change in the classroom: group 1 took a step back while group 2 took the leadership of the game organization. In a few days group 3 adhered. Along the study we noticed no deleterious attitudes. The following episodes reported in the teacher's diary illustrate how the pedagogical challenge was met.

Episode 1. There had been a party the day before and we were assigned to another room. The tables and chairs were all piled up against the walls. *Well*, I said, *I don't think that we will be able to play today, since the room is in poor condition...* Students protested: *Of course we can, teacher! Go with the two guys to get the game materials in the parking lot. We will fix the room.* When I came back they had already arranged the tables and chairs in the center of the room.

Episode 2. Children have been extremely careful not to lose nor damage the game' materials. They have been enthusiastic about the games. They asked me if they could make them to sell around their neighborhoods. Another reported to me that they were playing at home, with their relatives and friends.

Episode 3. The children introduced an extra rule for the game of bets: *the bet ought to be on the table before the card is drawn*. This is because some students can foresee their best bet from a glance at the card.

Episode 4. When the situation of figure 4 was represented in the worksheets, several students commented with amazement: "*Teacher, look: when the signals are equal it is plus, when they are different it is minus!*" I answered trying to appear very casual: "*Of course*".

We designed the game of bets because we found that the macaws game was difficult for us and for our colleagues with whom we made the first trial runs. However, the children reported that it was easier than the butterflies game. We suspect that they had already acquired the necessary degree of abstraction from the butterflies game, so that the macaws game looked easy.

The interest and class attendance persisted even after the games were finished and worksheets were introduced. Students solved the worksheet problems really fast and demanded new ones, almost exceeding our capacity to produce them. No extra explanations were necessary for activities that had no direct counterpart in the game diagram. We suspect that the students had already developed a certain ability to read, since we had insisted that they red the rules of each game before asking us to explain them. In some cases we even resorted to blackmail: *It is a pity that we can't go on playing, since you do not understand the rules...*

As for more facility of playing, no difference was detected between grades 5, 6 or 7. Children played equally well, they needed the same amount of explanations, they took approximately the same time to engage in the games and they requested help at the same points. The relevance of this outcome stems from the fact that official syllabuses recommend to start the study of negative numbers only from the 6th grade on. We had to work a lot but it paid off.

6. Discussion

The minus sign in the macaws has the property of changing colors. *Isn't this an arbitrary imposition of the sign rule ?* some people ask. Of course not. This is arbitrary but this *is not the sign rule !* Any game has arbitrary rules. But It is not when a negative multiplier acts on a colored bead that the sign rule comes up. The possibility of playing with operator qualities represented by signs and state qualities represented by colors definitely shows that this is not a commutative operation as in the sign rule. Performing "4-red times minus-three equals 12-blue", as in figure 4, amounts to the composition of a multiplication operator "three times" with a *change-colors operator*. The multiplication of negatives, as it stands in the sign rule, only becomes a commutative operation as the result of *composition of negative multiplication operators*, such as minus-two followed by minus three (see figure 4). It is the students who conclude that a plus-six card is required to close the circuit. This is the sign rule. *The sign rule involves two operators*, not an operator and a state. Besides, at this point, which of the two colors - red or blue - is going to play the negative sign is as yet undecided. It requires a lot of mathematical sophistication to imagine one of the colors as meaning "negative" and to see the phantasm of the sign rule in the change-color property of negative operators. Children do not have this sophistication.

It seems that we made the right decision in letting the children play for quite a long time before starting the activities. Surprisingly for us, they solved the activities quite fast and were engaged by them. What precisely have they learned with the games that made such a difference from the pilot experience the year before? We conjecture that the construction of integers "is an *operational synthesis* of multiplication and "change" that make operative and predicative signs merge together. Contrary to analytic processes, synthesis requires a deviation of attention from the object" [Baldino, 1998]. When the student comes to the point of asking *why does minus times minus equal plus?* it is already too late. S/he is centering attention on the question and the operational synthesis becomes difficult, or even impossible. On the other hand, it seems that the games have facilitated such a necessary decentralization of attention so that the whole set of operations was carried out in action, before they were made explicit. In particular, our expectation that the students would explain the sign rule to us was fulfilled, as illustrated by episode four.

As a final word, we should comment on the freedom we had to authorize ourselves to work with games regardless of what was prescribed in the syllabuses. Considering the school's situation, we decided that anything culturally engaging that we could do with these students would be better than any reverence to official façades. We do not regret it.

Here is our final question: How will these students do in algebra?

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MOMENTS OF CONFLICT AND MOMENTS OF CONVICTION IN GENERALISING

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This paper reports on part of a study of students' ability to handle algebraic generalisation problems. In this paper we focus and elaborate on moments when students grapple with deciding about the validity of their generalisations. We interviewed ten students near the end of grade 7. During the interviews we tried to create cognitive conflict by challenging the students' justification for the methods they used and then documented their attempts to resolve such conflicts. We found that most students' justification methods were invalid, because they are not aware of the role of the database in the process of generalisation and validation.

INTRODUCTION

Number patterns, the relationship between variables, and generalisation are emphasised as important components of algebra curricula reform in many countries and also in South Africa. Much research has been done on children's generalisation processes documenting children's strategies in abstracting number patterns and formulating general relationships between the variables in the situation (e.g. Garcia-Cruz and Martinon, 1997; Taplin, 1995; Orton and Orton, 1994; MacGregor and Stacey, 1993). Our own ongoing research confirms many of these findings.

However, little research has been done in analysing children's thinking in the *processes* of generalising. For example, do students view their efforts at generalising as *hypotheses?* Do they realise the necessity to validate their methods and answers? How do they become *convinced* of the validity of generalisations? Garcia-Cruz and Martinon (1997) for example, report that most children they interviewed checked their rules. This was done either by counting or drawing or extending the numerical sequence. It is not clear from their report, however, whether their students *spontaneously* checked their answers because they felt the need for validation, or how they became convinced of the validity of their strategies and answers.

In this paper we focus and elaborate on such moments where students grapple with deciding about the validity of their generalisations. During interviews with children, we tried in several ways to create cognitive conflict by challenging their justification for the methods they used and then documented how they tried to resolve such conflicts.

RESEARCH CONTEXT

The research reported in this paper is part of an ongoing research project aimed at informing curriculum development. The project enlisted eight schools in the suburbs of Cape Town as project schools. Seven of the eight schools are in traditional black townships. All the children interviewed in this research came from one of these seven schools. As a baseline study for the project's diagnostic purposes, we have been collecting data on children's performance in mathematics using various tools. One of these tools is a written baseline test.

RESEARCH METHODOLOGY

As a first stage we wanted to gather data on the most mathematically competent students. The students were chosen on the basis of performance in the baseline test and the teacher's evaluation. We interviewed ten students near the end of grade 7. Each student was interviewed three times in 45-minute sessions by two of the researchers, twice individually and once in pairs. A fourth session took place in which the students were given two generalisation problems to do individually. All interviews were videotaped. In addition to the video protocols, written transcripts of the subjects' verbal responses as well as their paper-and-pencil activities were used in the analysis.

THE PROBLEMS

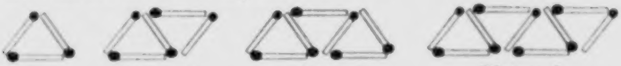
We presented the students with a series of eight generalisation problems in which we varied the representation of the problems. Some problems were formulated in terms of numbers only (in the form of a table of values), some were formulated in terms of pictures only (in the form of a drawing of the situation) and some problems were formulated in terms of both pictures and numbers.

The questions were in each case basically the same, namely given the values of f(1)[1], f(2), f(3), and f(4), we asked students to find the values of f(5), f(20) and f(100) and to explain and justify their answers and strategies. Six of the functions were linear functions of the form $f(n) = an + b$, and two functions were simple quadratic functions of the form $f(n) = n^2$. Here are two examples, "cans" and "matches":

(B1): Cans are packed to form pyramids.
The table shows how many cans are needed for different pyramids.
Complete the table.

Pyramid number	1	2	3	4	5		20		100
Number of cans	1	4	9	16					

(C3): Matches are used to build pictures like this:



Picture 1 Picture 2 Picture 3 Picture 4

The table shows how many matches are used for the different pictures.
Complete the table.

Picture number	1	2	3	4	5		20		100
Number of matches	3	5	7	9					

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[1] Formal functional notation was not used in the actual problems or in communications with the students. It is merely used here for reporting on the students.

Whatever responses the children gave, we asked them to *explain* their answers by posing questions like: "Can you explain how you got this answer?", or "Convince me that your answer is correct", or "Show me how you got this answer". If the students' explanations were based on the information given in the problem (the *database*), we accepted it as a satisfactory answer.

SOME RESPONSES

Some general observations

Concerning children's use of different representations for the problems, it is interesting that all but one of the children worked exclusively in the number context and did not use the structure of the pictures at all. In the problems that were formulated in terms of pictures, children immediately constructed a "table" of values and then used only the table of values in their solutions and explanations.

Concerning children's strategies, it is interesting that for the simple quadratic problems, nearly all the children recognised the functional rule $f(n) = n^2$ from the database and used it to find the values of f(5), f(20) and f(100). For example, Thandi explains: "I say 2 times 2 is 4, 3 times 3 is 9, 4 times 4 is 16, 5 times 5 is 25". However, in *all* the linear problems all students correctly used recursion to find f(5) as f(4) + d where d is the common difference between successive terms. For example, Thandi explains how she finds f(5) = 36 in the table for the function f(n) = 8n + 4: "I say 4 plus 8 is 12, 12 plus 8 is 20, 20 plus 8 is 28, 28 plus 8 is 36".

It is also interesting that when they had to find f(20) and f(100), most children abandoned their successful recursive strategy because they were trying to find a "shortcut" to calculate f(20) and f(100). These short methods were mostly *not* based on the database and were seriously prone to error. None of our students felt the need for any kind of validation. Although they offered some kind of explanation for the method they used in the extended domain, they were not aware of the role of the database in the process of validation.

Some alternative interpretations

In several cases children's answers were far from what we would expect, yet still based on the database. For example, in the cans-problem Roy wrote that f(5) = 23, f(20) = 28 and f(100) = 31, because he was using a symmetry structure (3; 5; 7; 7; 5; 3):

$$f(2) = f(1) + 3;\quad f(3) = f(2) + 5;\quad f(4) = f(3) + 7$$

$$\therefore\quad f(5) = f(4) + 7;\quad f(20) = f(5) + 5;\quad f(100) = f(20) + 3.$$

He ignored the shaded columns that we intended as representing several "missing" columns in the table.

Also in the cans-problem Sipho wrote: f(5) = 20, f(20) = 100 and f(100) = 600. This seemed to us rather arbitrary, but he was in fact using the rule that f(n) is a multiple of n, without specifying which multiple:

Interviewer: Can you explain how you got 20? [*for f(5)*]
Sipho: I took pyramid .... [*pause*] .... I saw that each doesn't have a remainder.

In the same problem Vusi wrote f(5) = 25. We were, of course, sure that he was using the functional rule $f(n) = n^2$. However, then he wrote f(20) = 120 and f(100) = 800, explaining that "you multiply each number in the upper row by the number of the column". Closer questioning revealed that he misinterpreted the shaded columns. For Vusi n = 20 was in the 6th column, so f(20) = 20 × 6, n = 100 was in the 8th column and his rule therefore produced f(100) = 100 × 8 = 800. He ignored the first shaded column and then counted the next 3 columns as the 6th-, the 7th- and the 8th column.

Some mistakes

Students made various mistakes, for example, to concentrate only on the relationship between a single pair (n ; f(n)) and then to use it as a general rule. For example, in the cans-problem Roy saw that f(3) = 3 × 3 = 9 and then used the rule f(n) = 3n to find f(5) = 3 × 5 = 15.

However, the most common, nearly universal mistake children made in their efforts to find a manageable method to calculate larger values, was to use the proportionality property that if $x_2 = k \times x_1$, then $f(x_2) = k \times f(x_1)$. For example, in the matches-problem, Mathole, after finding f(5) = 11, calculates f(20) as 4 × 11 = 44. This mistake was also found by Taplin (1995) and Garcia-Cruz and Martinon (1997).

CREATING CONFLICT

When we were not convinced that the students' responses reflected awareness of the role of the database in the justification process, we tried to create a cognitive conflict, using three different strategies as described below. (Because children were not using the pictures, we did not use a strategy of drawing pictures to check their answers.)

Strategy 1: The first strategy we used was to confront the answer driven from the recursive approach with the one obtained by the mistaken approach. This strategy was used when the child had in front of him/her a table he/she had formed in order to find some f(n) through recursion.

For example, Vusi and Thandi, working as a pair, used the recursive method to correctly determine f(20). In order to determine f(100), they systematically continued using the recursive method. However, when they reached f(50) they changed to the multiplication method, claiming that f(100) = 2 × f(50). We wanted them to reflect on the incorrect multiplication method. For this purpose we challenged them to apply their multiplication method on the domain between 1 and 50 since they had already obtained these values by the recursive method. Vusi was asked to find f(20) using f(5) and the multiplication method.

Vusi: Its 72 [*multiplying 18 by 4*]. I got 63 [*the result he obtained by the recursive method*].

Vusi is puzzled but still unconvinced that his method is wrong. He decides to recheck his multiplication method on the database:

Vusi: Lets try this one [*looking at f(2) and f(4) in the database*]. If 2 goes 2 in 4, so I must multiply 9 [*the value for f(2) in the given database*] by 2 is 18, but its 15 [*the value for f(4) in the given database*].

Interviewer: So what do you say when I ask you about 100?

Vusi: I said 20 times 5 so its 100. So 63 [*the value he obtained for f(5)*] times 5.

Vusi is sure that his answer for f(20), 63, he obtained by the recursive method is correct and the other answer for f(20), 72, obtained by 4 × f(5) is wrong. He is sure the method to get f(4) by 2 × f(2) is incorrect but at the same time he is not willing to give up his multiplication method when it comes to f(100).

Strategy 2: The second strategy was to create a conflict by choosing a take-off point different from the one the child had used when applying the multiplication method. Choosing different take-off points led to different answers for f(n). For example, Vusi spontaneously evaluates f(100) as 2 × f(50) = 2 × 147 = 294. The interviewer prompts him to use different take-off points. He takes f(10) and f(20) and obtains f(100) = 10 × f(10) = 330 and f(100) = 5 × f(20) = 315.

Interviewer: Oh, so who is right?
Vusi: Now we have three plans.
Interviewer: Ok, I understand three plans, but I also have three answers, 330, 315 and 294. Are they all right?
Vusi: Yes, they are all right.

Thandi and Vusi are sure about the values they obtained for f(10), f(20) and f(50) since these values were obtained by the recursive method. f(100), however, is an abstract entity for them. The fact that the three different take-off points led to three different answers for f(100) did not lead them to question the *method* they used.

Mathole, when confronted with different answers for different take-off points is also not prepared to abandon the multiplication method, but attempts to give a justification for the different answers:

Interviewer: And now you said that in shape number 20 we have 144 okay? Because you took this 5 ... you divided 20 by 5 and timesed 36 [*the value for f(5)*] by 4. We are sure about it. Okay, let's say that your friend goes to shape number 4 and he now divides 100 by 4. To divide 100 by 4 gives 5, so he goes and times 28 [*the value for f(4) in the given database*] by 5, do you follow me?
Mathole: Yes
Interviewer: So he multiplies 28 by 5, how much is it? [*works on calculator*] 140. So what is the correct one, 144 or 140?
Mathole: 144
Interviewer: Why?
Mathole: Because ... here by the fourth shape you got 28 matches and fifth shape is 36 matches, so if he goes back to the ... to the ... 28 he'll have to add 4 and if he goes back to the third shape he'll have to add 8, it's like you tax a person for going back, you let him pay for going back, so he'll have to pay 4 for going back, then you'll have to add a 4 there, then you'll get the 144.

Strategy 3: The third strategy was to implement the method the child used in the extended domain on the domain given in the original table.
For example, Thandi obtained an answer for f(5) by correctly using the recursive method f(5) = f(4) + 8 = 36. For f(20), however, she wrote 28, explaining:

Thandi: I count to shape 5, and I count to 20 and then I add this top numbers [*refers to the shape number in the table*] by 8.

While $f(5)$ was obtained correctly using the recursive rule $f(n) = f(n-1) + 8$, she now changes her rule to find $f(20)$ by using the function rule, $f(n) = n + 8$. She is then taken back to $f(5)$ and asked how she obtained 36. She adds 8 to 5 (the shape number) and gets 13, not 36. She now realises that there is a contradiction. Thandi now no longer accepts her answer for $f(20)$.

STRUGGLING FOR CONVICTION

It was clear that conviction about the role of the database in the process of validation develops *slowly*. Despite our efforts to create conflicts in order for them to reflect on the proportionality multiplication error and on the process of validation in early interviews, the same children repeatedly made the same mistake in later interviews. We follow below Sipho's struggle to come to terms with the proportionality multiplication error.

In the first problem given, Sipho obtained 36 for $f(5)$ by using recursion correctly. However, for $f(20)$ he abandoned recursion and used the multiplication method explaining: "5 goes four times in 20 so I multiply 36 [*the value he obtained for $f(5)$*] by 4 to get the number of matches in shape 20." The interviewer challenged him to apply his multiplication method on the domain 1 to 5, to obtain $f(4)$ as $2 \times f(2)$ and $f(5)$ as $5 \times f(1)$. Sipho was sure that the answer for $f(5)$ he obtained by recursion was the correct one and not the answer obtained by the multiplication method. However, when asked again about $f(20)$ and later on $f(100)$ he consistently used the multiplication method. This happened again in the next problem.

In the second interview Sipho was working with David. Both of them used recursion to obtain $f(5)$. However, for $f(20)$, David continued systematically with recursion, finding $f(20) = 63$, while Sipho used the multiplication method, finding $f(20) = 4 \times f(5) = 4 \times 18 = 72$.

Interviewer: I do not follow, shape number 20 is 63 or 72? [*strategy 1 as above*]

David: I go my way, adding 3 and 3 and 3

Sipho: [*to David*] I see the method is right, but can you tell me what I have done wrong to get the wrong answer?

At this point Sipho confronts the two methods which is significant since he realises that there is a conflict. He is sure the recursion method gives the correct answer and realises that his multiplication answer gives an incorrect answer. He is interested in *why* the multiplication method is wrong. However, in the very next moment Sipho again succumbs to the multiplication error:

Interviewer: What about shape 100?

Sipho: I times because you know that I get 5 20's .... I think I'll times 63 by 5 to get it.

David: That's the wrong way.

Sipho: If I didn't times, I added 3,3,3, ... I would get the same answer.

Interviewer: Where do you see multiplication? [*Strategy 3*] I can understand where the 3 came from. I saw that it's given here [*points at the table and the differences between the number of matches*]. Where did you get the multiplication? Can we check?

Sipho: Shape 3. I just go to shape 4.

Interviewer: If you want to get to shape 4 with your method what would you times?

Sipho: I would times the number of matches here [*points at shape 2*] by 2.
Interviewer: And what number of matches will you get?
Sipho: 18
Interviewer: And what is written here? [*points at f(4) in the given data base*]
Sipho: 15
Interviewer: So?
Sipho: Yes,eh

At the end of the interview we are left with the impression that Sipho is convinced that his multiplication method is incorrect, because the given database does not reflect the multiplication method.

In the third interview it appears as if the previous discussions with Sipho had not taken place. He still uses the incorrect multiplication method to obtain f(20). He is taken back to the given database to reflect on how he obtained f(5):

Sipho: Because shape 1 is 3 and shape 2 is 5 and the difference is 2.[*referring to C3*]

Sipho is now pushed to reflect on the given database and his method for obtaining f(20). He realises that if he uses the multiplication method on f(1) to obtain f(4), it will not be the same as the value for f(4) in the given database. This conflict leads him to use recursion to find f(20) = 41. Yet he reverts to the multiplication method to obtain the value for f(100). He is challenged by the interviewer:

Interviewer: It does not work for f(20) but you think it might work if you go from 20 to 100?
Sipho: Yes, because I think the number of matches in shape 20 is *now* right.

This remark sheds some light on Sipho's line of thought. He thinks since he *now* has the correct value of f(20) he can use it for f(100). For him, the problem was not the *method* but the wrong *value* of the take off point. It seems that Sipho is sure about the value of f(20) which was obtained by recursion. He is convinced that the multiplication method does not work for f(20), but nevertheless, from his perspective, it still works for f(100), provided that the value of f(20) is correct. He is now challenged to use the multiplication method on f(5) to obtain f(20). This yields a value of 44, which he knows is wrong because he obtained f(20) = 41 by recursion. He is puzzled:

Interviewer: Now, you think 5 times 41, 205, you say it's right for f(100).
Sipho: I think it's wrong.
Interviewer: Why
Sipho: Because I did the same thing when I multiplied. I tried to multiply the number of matches by 5 … I saw that I was wrong.
Interviewer: So how will you then do 100 [*shape number 100*]?
Sipho: I think I have to do it like this [*points at the list for f(20)*] but it will take a long time.
Sipho: [*Long pause*]…I'm trying to think if I can do another method to get the answer of eleven [*the answer for f(5) - he is trying to look for a functional rule*]. I'm trying to multiply the number of… number of matches in shape 5.

Yet, albeit a slow development, there were successes: in the final written test six out of the ten students avoided making the multiplication error.

DISCUSSION

Our study shows that our interviewees did not view their answers as hypotheses that should be validated. They were not aware of the role of the database in the process of generalisation and of validation.

Although we were aware that students frequently succumb to the proportional multiplication error, its persistence and obstinance to change surprised us. On the one hand students easily convinced themselves that when a value for f(n) they obtained using recursion differed from the value they obtained using their multiplication method (our validation strategy 1 described above), the result obtained by multiplication was *incorrect*. On the other hand, the knowledge that their multiplication method produced incorrect answers did not *prevent* them from making the mistake again (and again). Indeed, when they were asked for the value of f(m) for $m > n$ in the same problem, all the students again resorted to the multiplication method and were sure that their answers are correct. Our efforts to create cognitive conflict by leading students to apply the multiplication method to different take-off points and getting different values for f(n) (our strategy 2 above), or drawing their attention to the fact that the multiplication method is not applicable in the domain given in the table (our strategy 3), did not easily eliminate the error. Most students continued using the multiplication method throughout the interviews. Six out of the ten students eventually avoided the error in the fourth session.

One could argue that our choice of numbers *triggered* the proportional multiplication error, i.e. that our use of "seductive numbers" like n = 5, 20 and 100 *stimulated* the error. One could also argue that that if we used non-seductive numbers like n = 17, 27 and 83 children would not use the erroneous multiplication method. However, we believe that our evidence shows that children, in their quest for a manageable short method, *create* "seductive numbers" themselves. For example, Thandi was busy using a laborious recursive strategy on her way to calculate f(100) – she continued to f(50) and then suddenly stopped and calculated f(100) as $2 \times f(50)$, probably because she immediately recognised the multiplicative relationship between 50 and 100. Nevertheless, it remains a question for further research to establish whether an approach with non-seductive numbers will *prevent* children from making the multiplication error, also when they encounter seductive numbers in other problems.

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STUDENTS' REASONING ON QUALITATIVE CHANGES IN RATIO: A COMPARISON OF FRACTION AND DIVISION REPRESENTATIONS

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Two versions of a test on qualitative changes in ratio were administered to 318 Secondary Two students in Hong Kong. The tests were identical except for the representations of the ratios; x/y in one test and x÷y in the other. After analysis of the written tests 20 students were given detailed follow-up interviews. This paper discusses the findings from the tests and the interviews with a particular focus on the responses to the indeterminate items.

Introduction

Algebraically the two expressions x/y and x÷y are equivalent. Nevertheless, from a psychological perspective these two expressions 'feel' rather different and may well conjure up different images or connotations in the reader. For example, the first may simply be associated with a rational number, a fraction, that stands alone and needs no further comment while the second may suggest an operation that needs to be performed in order to find its value. Indeed, in learning about fractions, one of the concepts that children acquire is the fact that *one interpretation* of x/y is precisely x÷y, depending on the context. Although such expressions may variously be related to ratios, rates, proportions etc., for the sake of simplicity I shall refer to these expressions as the Fraction and Division representations of a ratio.

The purpose of this study was twofold. First, to investigate how children attempt to answer qualitative questions in ratio involving the direction of change. That is, given an expression like x/y, how does its value change when the values of x and/or y change? The second purpose was to determine whether or not there is any difference in performance on such questions between Fraction and Division representations. (Chung (1994) suggests this may be a factor). The possible changes in the value of a fraction (or division) given changes in x and y are shown in Table 1 below. This is based on a similar table for rates in Heller at al. (1990).

Table 1 *Changes in the value of x/y (or x÷y) as x,y vary.*

Numerator, Dividend	Denominator, Divisor		
	Increases	Unchanged	Decreases
Increases	Impossible to say	Increases	Increases
Unchanged	Decreases	Unchanged	Increases
Decreases	Decreases	Decreases	Impossible to say

Methodology and Results

From table 1 we can construct a set of 9 questions covering these outcomes. However, the case of x and y both unchanged is clearly trivial. Thus two sets of 8 multiple choice questions were produced, the two tests being identical except for the use of the expressions x/y and x÷y. An example of the wording of the questions is given below:

x and y are positive numbers. What happens to the value of x÷y if:

	increase	no change	decrease	impossible to say
a) x is increased and y is unchanged	☐	☐	☐	☐

The two versions of the test were administered to 318 Secondary Two students in 5 schools in Hong Kong; 160 received the Division version and 158 the Fraction version. Although all the schools were using English language text-books the tests were given in a bi-lingual version (English and Chinese) to avoid language difficulties. The results are shown in Tables 2 and 3, with correct responses in bold. The Increase, Decrease and Unchanged situations are indicated by the symbols ↑, ↓ and –.

Table 2 *Percentage figures for x÷y test. (N = 160)*

Question	Increase	No Change	Decrease	Cannot Say
a) x ↑ ; y –	**86.3**	2.5	6.3	5.0
b) x ↓ ; y –	9.4	0.6	**85.6**	4.4
c) x ↑ ; y ↓	**71.3**	9.4	7.5	11.9
d) x ↓ ; y ↓	16.3	33.1	18.8	**31.9**
e) x – ; y ↓	**73.8**	6.2	14.4	5.6
f) x ↑ ; y ↑	15.6	35.6	15.6	**33.1**
g) x ↓ ; y ↑	8.8	5.6	**75.6**	10.0
h) x – ; y ↑	11.3	3.1	**77.5**	8.1

Table 3 *Percentage figures for x/y test. (N = 158)*

Question	Increase	No Change	Decrease	Cannot Say
a) x ↑ ; y –	**86.1**	3.2	5.1	5.7
b) x ↓ ; y –	4.4	4.4	**86.7**	4.4
c) x ↑ ; y ↓	**77.2**	7.0	7.6	8.2
d) x ↓ ; y ↓	8.2	30.3	25.3	**36.1**
e) x – ; y ↓	**70.9**	6.3	14.6	8.2
f) x ↑ ; y ↑	22.8	34.8	6.3	**36.1**
g) x ↓ ; y ↑	6.3	4.4	**77.8**	11.4
h) x – ; y ↑	12.0	3.8	**76.6**	7.6

As can be seen almost at a glance, there are no statistically significant differences in the facility rates of any of the eight questions. Indeed, one is struck by the remarkably consistent distributions of the responses for most of the questions. This would suggest that for this type of qualitative ratio question the symbolic expression used (i.e. x/y or x÷y) has no effect on the students' performance. However, before jumping to this conclusion we should notice that these results tell us nothing about the *way* in which the students are arriving at their answers and, in particular, what is the nature of the difficulty they have with the 'indeterminate' questions (i.e. where x, y both increase or decrease). These two questions produced far lower facility rates (just over 30%) than any of the other items and the results are quite similar to those found by Heller et al. for 8th grade students in the USA (28.5% for the decreasing/decreasing problem). In fact, it is precisely for these two questions that we *do* find a striking difference between the distributions in the two tables. Although the facility rates and the responses in the No Change categories are very similar, we observe a curious phenomenon for the responses in the Increase and Decrease categories as highlighted below.

Test	Question	Increase	Decrease
x/y	d) $x\downarrow, y\downarrow$	8.2	25.3
x+y	d) $x\downarrow, y\downarrow$	16.3	18.8
x/y	f) $x\uparrow; y\uparrow$	22.8	6.3
x+y	f) $x\uparrow; y\uparrow$	15.6	15.6

The figures show that for all four of these items, about a third of the students gave either an Increase or Decrease response. However, in the x/y cases the ratio of the figures in the two categories is over 3 to 1 whereas in the x+y cases the figures are roughly equal (in one case, exactly equal). Choosing the Decrease answer in the case when x and y both decrease (and Increase when x and y both increase) could be described as an Intuitive response (i.e. simply reacting to the stimulus words). Similarly, we might then describe choosing the opposite category (e.g. Decrease when x and y both increase) as a Counter-intuitive response. Using this terminology, what we need to explain here is why there is a far heavier selection of the counter-intuitive options for the x+y cases compared to the x/y cases.

In order to answer this question in particular and also to probe more deeply into the students' reasoning in general for these types of question, twenty students were selected for interview (ten from each test). The major focus of interest was on their responses to the two indeterminate questions (d) and (f), and hence the students were selected on the basis of providing a representative sample of the different types of responses given to these two questions. However, despite this emphasis on the indeterminate items, the students were also asked to explain two other items and nearly all the interviews began with an item that had been answered correctly (although this was not shown on their papers). This helped to give a picture of whether or not a consistent strategy was employed.

The Interviews

A number of consistent features emerged from the interviews, the most striking being the use of a substitution strategy. Only 2 of the interviewees said that they did not 'try out numbers' but simply based their answers on feelings or guesses. A few of the students said that they used substitution for *some* of the items but they either guessed or 'knew' with other items. The rest claimed to use substitution for all the questions. In the written tests 7 students on each of the Division and Fraction tests (i.e. 14 in all) ticked 'Impossible to Say' for *all* the items. One of these students was interviewed and it seems likely that his explanation would be typical for this subset: "I think I do not know the numbers at the start. So when they change I still cannot know the numbers. So it is impossible to say". The significance of this rather special set of students is that, since they gave correct responses to the indeterminate items for clearly inappropriate reasons, the facility levels for these questions are in fact slightly distorted and should be reduced by 4.4%. (Of course it is possible that other students too gave the right response for the wrong reason). A typical example of using the substitution strategy is shown below (I = Interviewer):

I: Can you explain how you decide your answer to (c)? [*Fraction Test*]
WS: First I will try 10 over 5. Then this number increase by 2 to 12, and this one I will minus it by 2 to 3. This is 4 over 1 [*cancelling 12/3*]. So it increases.

Turning to the other student who did not use substitution, and using the terminology suggested earlier, she gave an Intuitive response for each indeterminate question, as illustrated in the following extract:

I: Here [*item (d) of Division Test*] you said the answer would decrease. Can you tell me why you put this?
AY: Because when I do this I .... I'm not really to solve it. I just tick the answer.
I: So how did you decide which answer to tick?
AY: Just looking. I didn't try numbers [*referring to a previous comment in the interview*] I think I just read 'decrease' so the answer must decrease.

This is a good example of a response based on the stimulus words in the item; the reason for labelling it 'Intuitive' in the previous section. However, since there were only two such instances of responding to stimulus words in the interviews, this suggests that the 'Intuitive' label for that category is not perhaps appropriate after all. Hence, in the following sections the Intuitive and Counter-Intuitive categories are re-designated by the more objective labels 'Same-Direction' and 'Opposite-Direction' responses. Let us now consider some of the student explanations within the different response categories. (As indicated earlier, these categories refer to the responses for the indeterminate questions).

a) No Change

Three of the students interviewed in this section initially chose equal values for x and y. For example, AM explains his answer to (d) [Division Test]: "I think x is 3 and y is also equal to 3. Both decrease by 1, so that is 2 divided by 2 and the answer is also 1. So no change". Another student, CS, began with unequal values and this is how the interview progresses:

I: Let's look at (d). Can you show me how you decided on this?
CS: Um ... I let x be 2 and let y be 3. Both decrease [*Writes 2/3 → 1/2*] This number is bigger [*pointing to 2/3*] and this is smaller [*pointing to 1/2*] so it is decreased.

I In the test you put No Change. Can you tell me why?
CS I don't remember.
I I guess you must have had a reason.
CS I think when I do this test I do it very fast. And maybe I think they both decrease so it is no change.

Indeed her original answer may well have been decided using such reasoning but, given her claim (in another part of the interview) to have tried numbers every time, this could be a post rationalisation. As with AM's explanation, it is possible that she also chose equal x, y values. Whatever the reason, this is an example of an 'unstable' response and similar instances occurred in a number of the interviews.

b) Same-Direction

A typical example of an exchange in this category is given below:

I Now let's look at (d) [*Fraction test*]. x is decreased and y is decreased and you said the answer decreased. Can you explain this?
WS I also try two numbers [*writes 4/12*]. Now decrease [*writes 2/10*]. I cancel this [*4/12 to 1/3*] and this [*2/10 to 1/5*] and now the numerator is the same and the denominator is larger [*pointing to 5*] so ... the larger is the smaller so the answer decreases.

On further elaboration it is clear that the phrase "the larger is the smaller' refers to a larger denominator resulting in a smaller fraction. As with the previous category, it is important to note here that the answer to a particular trial is dependent on two things: the original values selected for x and y, and the amount of increase/decrease chosen. This is discussed further in the next section. It is also interesting that one of the students interviewed in this group had answered the Division test but in the interview worked with fractions throughout and so gave a very similar explanation to that above.

c) Opposite-Direction

An illustration of this category is demonstrated by IR's explanation to item (f) [Division test]: "I let x a number and y is a number. Then I add 1 to x and also increase 1 to y. Like x is 10 and y is 5, x increase to 11 and y is 6. The answer was 2 but now it is less than 2, so the answer decreased". Here again we see clearly that the particular values chosen have determined the conclusion reached. Before turning to the crucial cases of the Correct response category, some further comment on these first 3 categories is warranted.

Discussion of Substitution Strategy

In using a substitution strategy, two other features were very evident in the interviews. The first was a strong tendency, when two changes were involved, to increase or decrease by the *same* amount. In very few cases were different increments used without prompting by the interviewer. The following extracts illustrate the students' justification for this:

I. I notice that you have always added or subtracted the same number, either both 1 or both 2. Do you think it *must* be the same number?
S. No, but I think when they are the two same numbers we can calculate easier.

WS [*in response to a similar question*] Actually I don't mind the numbers but I just chose the same here to make it easy

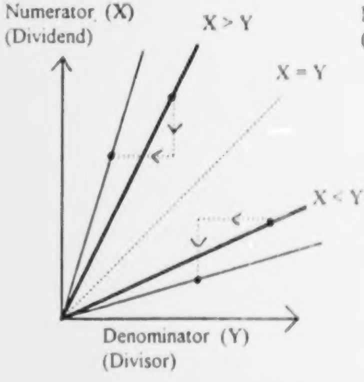
AM Yes. If you increase the number above you also must decrease the same number below
I [*later in same interview*] When it says x is decreased and y is decreased it doesn't say how much, so if I said x is decreased by 1 and y by 10 say, do you think that would be OK?
AM Mm ... Maybe ... but I don't like it

Ease of calculation was regularly cited despite the fact that in practice this sometimes led to difficulties in comparing initial and final values (e.g. when 6/8 was changed to 5/7). This reason was also quoted by the students who chose equal values for x and y where, of course, it did make the calculations easy. The second feature that was strongly evident relates to the relative magnitudes of x and y. In nearly all the x/y examples students chose x < y while conversely, for the x÷y examples the students overwhelmingly chose x > y. (This phenomenon is also reported in Barash & Klein (1996) where 58% of 7th graders believed that the dividend is always bigger than the divisor). Some typical comments are quoted below:

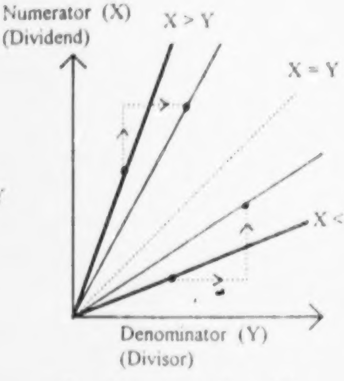
I: Was there any reason why you've always chosen x smaller than y?
T: Because it is a fraction so it is easier.
I: Why is it easier?
T: If this [*pointing to x*] is bigger than y it will be more difficult. [*Here it seems likely that T is associating improper fractions with 'difficult'*]

I: Was there any reason why you chose x bigger than y?
CH: Yes. Because it is x divided by y.

Both of these tendencies, equal increments and relative magnitudes, combine to give a clear explanation of the different distributions observed for the x/y and x÷y tests on the indeterminate items. For example, in the Division case, if equal increases are used when x > y this will always result in a decrease in the value of x÷y (and vice-versa for equal decreases). Lopez-Real (1997) shows the power of a co-ordinate representation for comparing ratios and we can illustrate the results here in the same way. (Note that due to the order of x, y given in the original tests, the labelling of the axes is the opposite of the conventional form).



Equal Decrease in X and Y



Equal Increase in X and Y

Correct responses

One of the features that characterised the students in the previous categories was the belief that a single trial would suffice to decide each case. Many stated that they were sure of the answer after just one substitution (even in the indeterminate cases). With respect to this, CS makes a revealing comment: "If there is time I will try some other numbers. Just to check". The final phrase suggests that CS is not thinking of different possible *outcomes* but rather as a check on her computation. In contrast, although some found it difficult to articulate their reasons, the common distinguishing feature of the students with correct responses was their *awareness* of these different possible outcomes. For example, AD interpreted all her answers in terms of apples and children and this is her explanation for (d) [*Division test*]:

> Because if there is 10 apples and 5 children then each get 2. Then if the apples decrease to 8 and there is 4 children they also get 2 each. But if the apples decrease to 9 and the children becomes 3 then they will get 3 apples each which is more. So ... I can't tell which will happen"

The main point of interest with the correct-response students is how they decide when it is necessary to consider more than one substitution and when not. In the following extract KW attempts to explain this. She has just successfully explained (c) by a single substitution.

I: Let's look at (d) now [*x/y type*]. Can you explain this?
KW: Say x is 4 and y is 5. If x decrease this can decrease to 3 or to 2 and if y is 5 it can decrease to 4 and 3 and 2. If it [*y*] decreases to 2 then this is bigger than this [*comparing 3/2 to original 4/5*] but if this decrease to this [*writing 4/5 → 2/4*] then the answer is decreased so I say 'Impossible to Say'.
I: Did you try more than one set of numbers each time?
KW: Just in the "Impossible to Say' I try more.
I: So here [*indicating successful answer to (c)*] why did you only try once?
KW: Because in my memory I knew this happen. From my lessons. [*she goes on to give a similar explanation for (h)*] .... If the denominator is more, then the fraction is smaller. We learn it already.
I: How could you explain to a friend that you need to try more than one set of numbers in some cases but not in others?
KW: Well, here [*pointing to (c)*] however x is increased .. by how many ... and y is decreased whatever much ... it doesn't matter, the answer will always increase. But here [*indicates (d)*] I ... um ... I can't catch the answer because it can be different.

Her confidence in the questions where she tried just one substitution appears to be rooted in her previous knowledge (e.g. "in my memory" "we learn it") and in this sense it is probably not even necessary for her to substitute at all. (A similar point is discussed by Mok, 1997). It may well be that giving an example to the interviewer is simply a way of explaining the answer. However, she still finds it extremely difficult to articulate why she knows certain items will be indeterminate and her frustration is beautifully illustrated by the phrase "I can't *catch* the answer".

Final Comments

The interviews clearly illustrate the dominant role of the substitution strategy in answering such 'directional' questions. In all of the items, except for (d) and (f), such a strategy will be successful with a single substitution, regardless of the values chosen for x and y or the increments chosen. However, for items (d) and (f) the specific values chosen are a determining factor in the outcome and this helps to explain the very different distributions for the x/y and x ÷ y tests for these items. It also means that a student's response is likely to be 'unstable' and a different response may well be given on another occasion (as was the case for some of those interviewed).

The most difficult question to answer is precisely how the *awareness* that certain items are indeterminate is achieved. The interviews with the successful students show that they are able to *explain* why their choices are correct but they still find difficulty in articulating how they know *beforehand* that items (d) and (f) require more testing. We can speculate on this by focussing for a moment on one of the other items. In fact, there is a certain circularity involved in a question like 'what happens to x/y when x is unchanged and y increases?' If one checks by substitution, say 2/3 changes to 2/5, then one already needs to *know* that 2/5 is less than 2/3 in order to give the correct response. In that sense, the property that the item is testing is already *embedded* in the particular substitution used. Now it is clear that most of the students did have this knowledge and hence were successful with that item. But perhaps the difference between such students and those who were also successful with the indeterminate items is that the former use the knowledge in a limited algorithmic sense (i.e. being able to check that 2/5 is smaller than 2/3) whereas the latter have incorporated it into a qualitative schema that can deal with variation independent of specific examples. However, it is clear that further analysis is required so that a more detailed picture of such mental schemas can be identified.

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Non-elementary Mathematics in a Work Setting

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We report on one strand of ongoing research into the use of mathematics in a computer aided design (CAD) and manufacture (CAM) setting. We focus on technicians' calculations of the volume of a mould they produce for glass factories. The 4-parameter model of Saxe (1991) forms a background for discussing observations, which suggest that the technology governs the mathematical reasoning they employ.

Introduction

This paper forms a self-contained part of a larger study concerning interrelations between geometric reasoning in various settings. Here we focus on the mathematical thinking of a small group of technicians who produce moulds for glass factories. Before presenting details we outline the wider aims of the research. In the larger study we consider geometry-related practices in a vocational school for machine technicians in mathematics classes and in 'application' classes (related to CAD) and professional CAD-practitioners in a work setting. Our basic research interests are concerned with the role of school geometry, as it is learnt in vocational schools, in work settings. Do practitioners (technicians) that come across geometry related problems revert to school geometry, do they feel that their work is related to school geometry, do they use school-learnt strategies, do they feel a discontinuity between school-geometry and their working practice? Like Noss & Hoyles (1996) we are interested in how they construct meanings to their mathematics-related actions and whether these meanings are learnt at school, on the job or a mixture of both?

Such questions are not new to research about mathematics in the workplace. Most of this research has shown a strong discontinuity between school and working practice. According to the situated cognition paradigm, e.g. Lave (1988), this discontinuity is a consequence of the fact that learning in a school environment and working on a shop-floor are two different social practices. Further to this, school mathematics is often ill suited to working practices. In some cases the problems are only apparently similar to school problems and in reality there is a range of explicit and implicit additional restrictions which makes school methods unsuitable, and thus other methods are used (see, for example, Masingila (1996)). In other cases, e.g. Scribner (1984), the tasks are only apparently mathematically simple and in reality there are no simple algorithms or methods to solve the problem and school-learnt procedures are of no use. To advance work in this area we wanted to direct our research to working practices where school-related knowledge is appropriate and could be profitably used, and where solutions could not be easily invented. Such cases are admittedly rare, but one example is the design and manufacture of moulds for bottles of irregular (non-standard) shapes in the glass industry by technicians with vocational school training.

Setting and methodology

The research took place at a small factory in Slovenia that produces moulds for glass factories. Six practitioners, all with vocational school training for machine technicians, were observed for 3 weeks (about 60 observation hours). Three of them were constructors[1] working with a 2D-CAD system (they have only recently introduced a 3D-solid modeller), the other three were technologists[1] working with machines for producing the moulds. They all used computer programs for the production of the moulds on numerically controlled machines. The observer was well acquainted with all the technology involved.

A variety of methods were used to elucidate the mathematical thinking of the observed practitioners: interviews, observation (135 mathematically relevant events/notes), scheduled observation, document analysis, interviews and protocol analysis related to given problems. We interpreted the data using the 4-parameter model of Saxe (1991), i.e. we interpreted the data in terms of activity structure, previous knowledge, social relations and artefacts used.

In this paper we restrict discussions to ideas related to volume. In the next four sections we summarise our observations along the lines of Saxe's 4-parameters. In the final section we discuss the relevance of these observations in understanding differences between school mathematics and (this) work mathematics. We conclude with an hypothesis about the practitioners' reasoning in this context.

The concept of volume is 'dynamic' and embodied in the activity cycle.

The practitioners use a variety of methods to establish the volume of a designed bottle. These methods are sometimes approximate and sometimes not, but for the practitioners this is not relevant, in practice they are all approximate with respect to the final product. In the production process the volume is calculated and measured several times with different (mathematical) methods. There is no point in thinking of the exact volume of the bottle (more precisely of the volume of the mould for the bottle) because the mould is subject to various approximate calculations and mechanical manipulations. As the picture shows the mould essentially consists of 3 parts: two lateral, usually symmetric, parts and the bottom socket.

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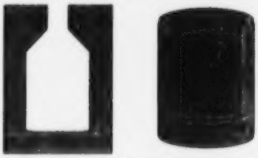
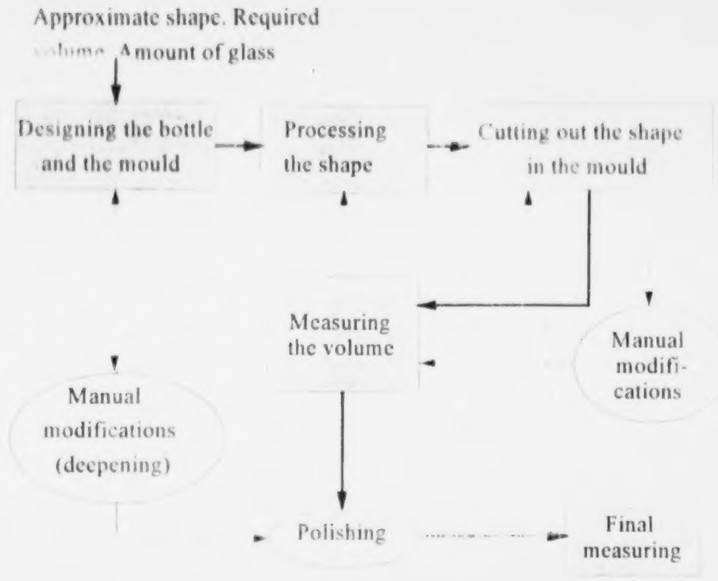
[1] Constructors basically evaluate whether a mould for a conceived bottle can be manufactured, define the dimensions of the raw mould (from which the desired shape of the bottle is cut off), design the bottle and make all the necessary technical drawings. Technologists generate programs for numerically controlled milling and other machines. They also define the surfaces to be cut and determine the volume. The practitioners enjoyed good relationships, mutual understanding of their roles and respect for each others work.



In a typical observed situation the practitioners have been involved in the construction of the mould for a bottle of a given shape, usually only approximately defined, made from a given quantity of glass, and having a given volume when filled up to a certain height. A constructor first designs the bottle and the required mould. The design is then given to a technologist who, using appropriate software, defines the procedure to work out the mould (s/he also recalculates the volume). Most of the cutting process is done automatically on numerically controlled machines, but small details are often worked out manually (which obviously changes the obtained volume). The volume of the mould is then measured by filling it with water. The measured volume has to be a bit smaller than the required one for it is normally enlarged by polishing the mould, a manual operation which increases the volume. If the measured volume is too small there are several ways of increasing it. The preferred way is to slightly redesign the shapes by increasing the overall size of the designed bottle – this method is preferred because it preserves the shape. Another way is by uniformly deepening the cut, i.e. manually deepening part of the shape. In general the technologist and the constructor consider the shape of the bottle, the allowed volume error and certain technological aspects and do not find it hard to decide which method to choose. But every method is costly in terms of time, so they do their best to obtain the 'correct' volume as soon as possible. However, if the measured volume is too big then the practitioners are in serious trouble. Sometimes the volume can be reduced by grinding out the central vertical part of the mould, but in general a costly mould is wasted.

Note that the volume is calculated or measured several times during the whole process by different people with different methods. As we shall see the methods are usually approximate (and the practitioners are aware of this). Also, there are one or more phases which cannot be brought under complete (mathematical) control by the practitioners. This aspect of the process is depicted in the diagram on the next page.

In the production process the volume is calculated and measured several times (at least four) by at least three people with different methods. Since all calculations are approximate and since some operations are always done manually there is no point of speaking about the exact 'volume'. In practice, the calculated volume is something **negotiated** among the participants. For example, if a constructor designs a shape of a nominal value (the calculations are regularly checked by another constructor) and a technologist calculates a different value (with another software program) then they have to agree on what to do. If one of the volumes is too big then they are much more careful for this is something to be avoided at all costs.


Approximate shape. Required volume. Amount of glass
Designing the bottle and the mould
Processing the shape
Cutting out the shape in the mould
Measuring the volume
Manual modifications
Manual modifications (deepening)
Polishing
Final measuring

Schematic presentation of the volume-related operations of the observed process of design and manufacture of the mould for the production of bottles. Shading of the blocks indicates the calculation or measurement of the volume; the elliptical shape of the block indicates that the operation is not under 'mathematical' control. The dotted lines represent decisions practitioners try to avoid.

We have not observed a preferred method for calculating the volume. The constructors and technologists do not discuss the validity or correctness of the methods used to calculate the volume. Their main concern is how to obtain the final shape with a given volume (though privately some claim that their calculations are the most accurate). We observed several methods for the calculation of the volume of a shape. Please note the central role of technology in their reasonings.

- To obtain the volume of a rotational shape, or part of a shape, the constructors draw the 2D-profile on a computer system and then use a program which automatically calculates the volume of the rotated shape. They were very careful when using this method because they knew that the software sometimes did not give the correct answer (the software requires that certain conditions hold for the shape and the way it is positioned and they prefer to check the result by another method rather than attend to the input conditions). The method has been shown to

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the constructors by the workshop manager and they knew that it used a 'correct' formula, i.e. it is not an approximation.

- The constructors often represent the shape of the bottle in terms of horizontal cross-sections at various height levels and draw a sequence of cross-sections using standard drawing procedures. These horizontal cross-sections are often just approximate, e.g. they draw a circular arc instead of an elliptic arc. The constructors are well aware of this. The cross-section is always composed of circular arcs and line segments, since the software calculates the area on this basis. The volume of the part of the bottle between two horizontal sections with respective areas A and B and the height h between the sections is calculated using the formula $vol=h/3(A+\sqrt(AB)+B)$. For a given bottle several such sections are drawn and the complete calculation is performed using a spreadsheet (if the volume is not correct, then aesthetic changes are made to the cross-sections until a desired volume is obtained). The constructors were not able to say who told them to use this formula - they claimed that it was 'a shop-floor tradition'. They are well aware that it is an approximation, but they did not relate it to the volume of a truncated prism or to Simpson's integration formula. A constructor explained that the volume of part of the bottle between two horizontal sections could be approximated by $vol=h\times(A+B)/2$, which, he claimed, is essentially the volume of the prism. He said that the formula they use is just a better approximation.
- The constructors have only recently obtained a 3D-solid modeller but during the observation period they almost never used it and produced all the documentation (for the technologists) with a 2D-drawing system. We once observed the design of a bottle with this modeller, in order to calculate its volume, but all the documentation was produced with a 2D-modeller.
- The volume of standard geometric shapes are calculated using school-learnt formulae, e.g. to calculate the volume of a prism the constructor drew its base on a computer to obtain its area and then used the formula $V=h\times A$.
- The technologists obtained the volume of a shape using a 3D-surface modeller. They reported that the accuracy of the calculation of the volume increased with the density of the grid on the shape (the program they use calculates the volume using polyhedron with the vertices on the mesh points, but the technologists ignored this).
- Once the mould was made, its volume was measured by weighing the water it could hold.

There appeared to be no order of precedence amongst these methods, they were used interchangeably and several different methods were often used for calculating the volume of different parts of a single bottle.

Understandings

All the practitioners appeared to have complete insight into the whole activity cycle, and they had a very good understanding of the reasons for and the role of the calculations they performed. They were able to explain every detail about the problems related to the volume but they paid little attention to the underlying mathematics of the methods used – they simply accepted them and tried to determine when they work but not why they work. One of the technologists, for example, was asked, 'Why not increase the amount of glass when the resulting volume of the mould is too big?'. He promptly explained the reason in terms of the machine that used the moulds to make bottles. All our observation suggest that all parties appreciated the mathematics they were using and could, when appropriate, relate this to mathematics learnt at school. It is important to stress, however, that although mathematics did give them a sense of what they were doing, they only seldomly used school-learnt methods and, when reasoning about volume, they did not reason in terms of mathematical concepts or methods but, rather, in technological terms (see below).

A related and regularly observed feature was using technology to 'correct' mathematics. Errors in volume calculations, due to inaccuracy of the mathematical methods or to inaccurate descriptions of the shape of the bottle, were corrected with technological procedures. If the calculated volume was too small the socket was made deeper or the whole shape in the mould was made a bit deeper. In defining complex shapes on a computer mathematical (geometric) errors often occurred. When this happened no attempt was made to analyse the reason for the error, they either immediately skipped to another mathematical method or left the problematic part to be completed manually. Certainly an important reason for such behaviour, although perhaps not the main reason, was the pressure of time. The main reason is, we posit, that the objects of their thinking are not mathematical objects but technological operations and artefacts which allow technological manipulation.

The role of the tools

The tools used, software and machine tools, significantly influence ways of working and of reasoning. Further to the discussion of the role of technology above we can illustrate this with two examples. In designing the cross-sections of the bottles (used for the calculation of the volume) the technologists use only circular arcs and straight lines. They deliberately avoid using splines[2], though they know about them, because they need the area of the section and the software does not calculate the area of a region bounded by splines. Thus, although they are aware that a curve is elliptic or irregular they deliberately draw it as a circular arc. Secondly, the software used automatically converts ellipses into a sequence of circular arcs. The practitioners are very confused about this and sometimes treat an elliptic arc as an ellipse and sometimes as a sequence of circular arcs (avoiding consideration of whether the arcs are an approximation to the ellipse or an exact construction).

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[2] Splines may be thought of as the curves used in computer paintbrush programs. Imagine a flexible strip of plastic passing through nails on a board. The shape taken is a spline. Modern dictionaries of mathematics or computer science will provide mathematical definitions.

The role of social relationships

Social relationships in the workshop play an important role in reasoning about volume. As we have mentioned, calculated volumes have to be agreed between participants. Also, there are mutual checks of each other's calculations in, as we have observed, an atmosphere of mutual respect. We perceived, however, that pressure from the workshop manager to get the job right first time, or a least with a small number of corrections, exerted considerable influence on the calculations. The constructors and technologists clearly respected the manager but, nevertheless, felt under great pressure from him. They would have preferred to produce a trial object, or to produce the final object, initially with a smaller calculated volume, which could then be increased. Although the manager agreed with this in theory, he dismissed such possibilities in practice.

School mathematics and (this) work mathematics

At one level the observed practices of the constructors and technologists in designing and producing moulds for irregular shaped bottles used aspects of school mathematics. Both integrated school mathematics into their working practice and the mathematics they used was meaningful to them – not in the sense that they understand all the procedures used but in the sense that the aims and the observed mathematical results were meaningful to them. On the other hand there was an evident discontinuity between the school mathematics used and the observed mathematical practices. This discontinuity was evident at both a subjective and an objective level. Although this discontinuity can be explained with reference to different cultures (classroom and job culture with the whole range of adopted specific methods, relationships, tools, activity structures) we would like to explore their use and view of school mathematics and links with technology.

At the objective level it was evident that the practitioners used school mathematics as a closed body of knowledge, as something that was frozen either in handbooks, in adopted formulas and procedures, or in software. This knowledge was used 'as is', without questioning it and without discussing or modifying it. In cases where a method did not work for some reason or an error occurred the practitioners regularly skipped to another method or found a technological solution to avoid a mathematical method that did not work. The mathematics they were really doing, their work mathematics, was inextricably joined with the technology they were using. The geometry elements always had a technological meaning – in this sense all the calculations they were doing were meaningful to them.

On the subjective level the discontinuity between school and work mathematics was again evident. The practitioners claimed, a similar observation is reported by Strasser & Bromme (1992), that there is no mathematics in their jobs. One practitioner, for example, reported:

Essentially I learn everyday on the job. I have completely forgotten about school.

But I see that in this theory, for example, this here (points to the splines on the display), theory and practice - it is very difficult.

If you do something in theory and then you try in practice, it never turns out like the theory predicts.

Recently, with these computers, they (theory and practice) are getting closer together. I see more and more now that what I've drawn here (on the computer) is exactly what comes out there (on machines).

The same practitioner was asked to solve some job-like problems. The protocol of his solution process shows that he clearly distinguishes the two mathematics. Here are some excerpts from his solution processes:

...I could measure this distance here ... though here mathematically I lack a datum...

...I could prove how to draw this purely mathematically. On the other hand, I could also draw this in an exact (i.e. non-mathematical) way...

...I would lose time finding a mathematical solution... I would not calculate it mathematically, I would get the position like this (successive approximations)...

...this could be done purely mathematically...

...I could draw this square mathematically or with some trials...

The observed difference between (this) work mathematics and school mathematics is thus, we argue, not just a matter of different contexts. An important difference was that in doing work mathematics the practitioners did use the school mathematics, but as a closed body of knowledge. Moreover, technology was such an omnipresent feature in their thoughts that we claim their mathematical reasoning is essentially technological reasoning in a mathematical context.

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On the difficulties of visualization and representation of 3D objects in middle school teachers[1]

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We report some results concerning reactions and difficulties met by middle school teachers involved in innovative activities of 3D geometry which require the ability to visualize the effects of some shiftings of solids or to evoke the vision of objects from particular points of view for their representation on isometric paper.

Introduction

It is well known that the teaching of solid geometry is rather neglected in many countries, and that is often limited to the calculation of the measure of surfaces and volumes (Howson e Wilson 1986, Howson 1991). Many authors, such as Bishop (1980, 1983), Gaulin (1985), Cooper (1986), Presmeg (1986), and Parzysz (1991) have underlined for a long time already the importance of promoting in the teaching of geometry the development of students' spatial intuition, including their ability to visualize and to transform visual representations and visual imagery. More recent studies (Mariotti 1989, Clement & Batista 1992, Presmeg 1994, Kopelman & Vinner 1994) have highlighted some of the difficulties linked to these activities, which even teachers have; some authors state that the pupils' difficulties mainly depend on a lacking classroom activity (Presmeg & Bergsten, 1995).

This study was started within a yearly seminar for teachers training devoted to solid geometry, and it was carried out with the belief that only by giving the teachers an opportunity to discuss and compare their problems and difficulties is it possible to make the necessary changes in classroom activity.

During the seminar we carried out an analysis of the contents of such topic in compulsory school syllabuses of various European countries, and we also examined some specific teaching projects. We focused out attention onto the English project *NMP Mathematics for Secondary School* (Harper 1987), in which much attention is paid to the study of solids, right from the beginning of first grade, through activities of various kinds which are suggested through very attracting worksheets, quite unusual in our teaching tradition and regardless of questions of measure. Such proposals, beyond the actual construction of objects with various materials, concern: 1) observation and description of the solids; 2) decomposition and recomposition of solids, representation and classification of their developments; 3) visulisation of solids from different points of view and their representation; 4) section of solids, even round-shaped, according to planes (not necessarily vertical or horizontal).

Despite their acknowledged charm, the study of these activities provoked the teachers' uncertainty about the difficulty of foreseeing the insertion of some of them in their lessons and some doubts on the hypothetical difficulties that pupils could have in facing such problems, also with reference to the level of schooling for which they had been conceived. In order to assess the suitability of such proposals, as well as to achieve an appropriate awareness of their aims, difficulties and potentialities, we tried solving them with a group of 8 teachers (women), who actually had considerable difficulties with them.

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[1] Report realized in the realm of MURST (40%) and CNR (contract n. 96.00191.CT01)

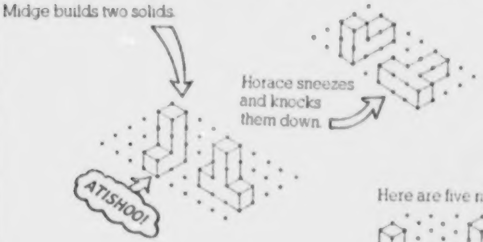
In this paper we refer to a part of this experience concerning the results obtained as to some problems created for 11-year-old pupils (first grade of middle school), concerning the representation of objects on isometric paper and under given conditions. These exercises imply the ability of visualising mentally the configuration of the solid in new positions or from a new point of view, as well as of correctly drawing such representation. We are going to present the problems examined by describing the aims and the foreseen difficulties, then we shall consider the teachers' productions, by analysing the errors and difficulties met. Finally, we shall draw conclusions on the basis of some considerations arisen during discussion.

The problems tackled.

The problems studied are reported in table 1, arranged into point (a), (b), (c) and (d). In (a) we show two problems in which it is asked to represent the solids in a different position from the one assigned, according to their falling towards a given direction. Such problems require the performer's capability of visualising mentally the object's configuration during their falling and of fixing such effect in the mind. The five objects (A, B, C, D, E) to be represented in the first question (which is preliminary to the second one) contain increasing difficulties; in particular, the representation of object D is difficult owing to the presence of 'steps', whereas that of object E is difficult because of the different width of its basis as to the other objects. The second problem, on the other hand, is much harder than the previous one, either for the complexity of the object to be represented (which can hardly be controlled globally in one's mind), or for the different falling direction, for which no example is offered. In point (b) we report two problems concerning the representation of objects reflected by a vertical mirror. The first problem asks for the representation of a 3D image of the letter "J" on the basis of a given example; the second one, that we created ourselves, asks to represent a certain object starting from its mirror image. The difficulties in carrying out these operations are manifold. Both problems contain the difficulty of visualising mentally the objects to be represented. In particular, in the second problem, it is difficult to imagine the back of the object reflected, which in the requested representation turns out to be at the front; then there is the difficulty of finding out the position of Alice. Other difficulties are mainly due to the need of defining some principles of representation for the sides that lie on parallel planes, and and in particular to overcome the difficulty constituted by the realisation of the images of those points of the representation of the object, each of which indicates points placed on different planes of the original object. In (c) we present some problems in which it is asked to complete the representation of some objects where parts of the outer surface are missing. The peculiarity of this problem lies in the fact that the last two figures could give vent to representations of different objects, which therefore requires the performer's ability of imagining the possible objects having such shape in the representation. Finally, in (d) we show a problem requiring the mental reconstruction, and consequent representation, of the vision of a rather complex object from a given point of view, which should be carried out on the basis of its representations from three different points of view. The main difficulty here consists in working out the mental vision of the object from the requested point of view according to the suggestions given.

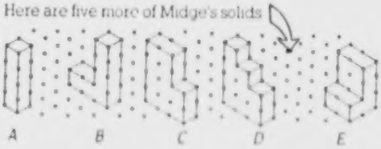
Table 1

(a) Representation of objects in a different position from the one assigned

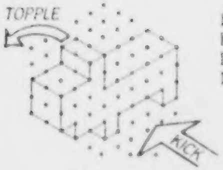

Midge builds two solids.

Horace sneezes and knocks them down.

Here are five more of Midge's solids

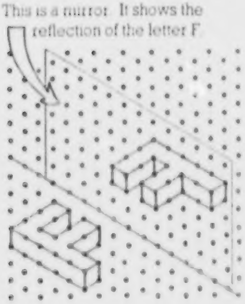


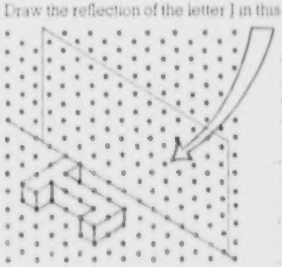
Draw what they look like after Horace sneezes

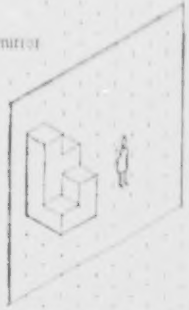


Lenny thinks this is a lorry
Fiona thinks it's a corgi
Lenny kicks it over
Draw what it looks like lying on its side

b) Representation of the image of objects through a vertical mirror.


This is a mirror. It shows the reflection of the letter F


Draw the reflection of the letter J in this mirror

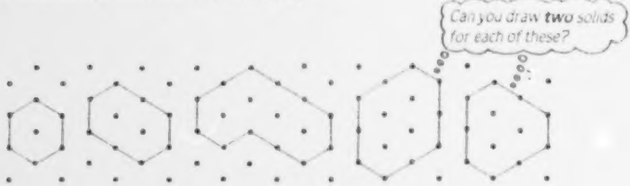

This is Alice "in" the mirror. Draw Alice and the tower autside the mirror

(to be continued)

(to be continued)

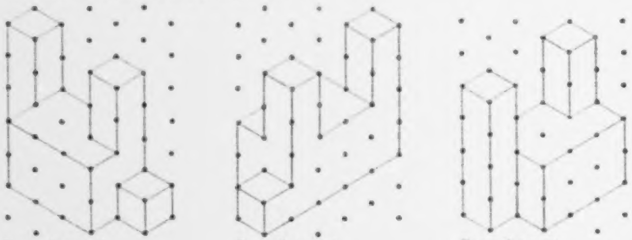
(c) Identification of objects and completion of their representation

Here are the outer lines of the drawings of some solids.
Copy the drawings, and draw the missing lines



(d) Mental reconstruction of the vision of an object and consequent representation

Here are three views of a power station



From the north From the west From the east

Draw what it looks like from the south.

The teachers' productions

To begin with, we would like to underline the fact that despite our request to record the sequence of attempts through which they get to the solution(s), the production didn't show - at least at the beginning - the immediate choices but rather the result of a sometimes very long activity of re-arrangement of failed attempts after initial block. As a general fact, we noticed that the teachers were reluctant to admit that they initially stuck and to show their progressive attempt (some of them even refused to deliver their papers, saying they would do it later on). Only after several working sessions, during the general discussion on this part of the activity, did the teachers declare spontaneously the difficulties they had and the long time they had devoted to the study of some of the problems.

The teachers' writings about the experience highlight the following difficulties with reference to all problems:

- to co-ordinate the partial visions of an object which must be represented in a different position from the one assigned as a whole, which was due to the prevalence -in the representation given - of images showing the visible parts of the objects instead of the hidden parts;

- to visualise the objects globally and fix them in one's mind in the right position or from the point of view requested for the representation;
- to evoke the vision of an object from one of the four fundamental points of view (north, south, east, west) from the representations of its visions from the other three points of view;
- to check the correctness of their productions and conceptualise the principles of representation (the operating rules are not explicit and should be discovered during the activity).

The progressive awareness of their own difficulties lead the teachers to the idea that if they wanted to promote this activity in school, a preliminary stage was needed, in which the pupils could have the possibility of building up such objects concretely, observing their real position from different points of view as well as their positions after falling towards various directions, and also learn to represent objects onto isometric paper.

Let's go deeper and analyse the teacher's answers, productions and reactions to each problem.

As to the first problem reported in (a), the teachers worked with good confidence, except for a few of them who had some difficulties with object D, because some points in its after-falling representation (the vertexes of the steps) actually represent two different points lying on different planes and some parallel segments in the representation (the edges of the steps) appear on the same straight line. They also had some difficulties with the representation of E owing to the fact that the basis appears in a higher position than the rest of the object. The main reasons for these difficulties are related to their not being much accustomed to using isometric paper (in fact some of them preferred using white paper). The second problem turned out to be rather difficult: some teachers stated their inability to visualise the position attained by the dog after falling, some said they could only see fragments of the representation, some other just stuck in front of the problem. The general attitude in dealing with the representation was to go through attempts and errors: and indeed, after starting with white paper the teachers did resort to isometric paper since they realised it helps in preserving length and parallelism.

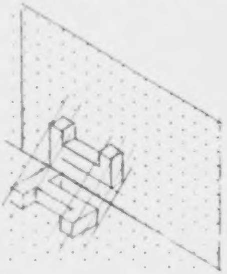
In the following meeting the teachers showed their representations, which in the end were mostly correct, though bearing evident spurs of corrections. In collective discussion on the difficulties and on the work carried out, the changing in thickness turned out to be a recurring error (most of them considered it of one unit first, and then of three units).

All teachers agreed about the difficulty of visualising globally the image that must be reproduced, but most of all of keeping (maintaining) such image, which made them refuse to chase the global mental image for the representation in favour of a "local" strategy for developing the representation by a simultaneous control of the mental image of the part examined. Only two teachers looked for a rational strategy focused onto the properties that bind some elements of the image (parallelism, perpendicularity, lengths, etc.) in search of a way of translating such relations into the representation, which in the end allowed them to find the correct solution.

The problems reported in (b), concerning the reproduction of images trough a vertical mirror, seem to be very difficult, especially the first one - not because of the shape of the object, but because of its position as to the mirror.

Again, the procedure to follow was to go through attempts and errors, with the prevalent intention of reconstructing the image globally, without controlling the actual equidistance of some key-points from the mirror, and which planes these points belong to. You get to the solution of the problem as soon as you realise that in the "J" shape to be reproduced there is a unite point that represents two different points: the first one belonging to the plane of the basis, at the distance of a unit from the mirror, the second belonging to the upper plane, at the distance of 2 units from the mirror. The most successful solving strategy was, here too, to start by representing the image of the points belonging to the plane of the basis, and only then to represent those belonging to the parallel plane at the distance of a unit higher.

Among these productions, there's a rather remarkable one that shows in the mirror the image in a very strange position (see picture). The thing is quite surprising, also because its author was rather sure about the correctness of her work. The reason for this error becomes clear as soon as we realise that the same image appeared in some of the pupil's productions.



The reason is that the figure is dealt with as if it were plane, and its correspondent is drawn according to the plane axial symmetry where the axis is the intersection beetween the mirror and the basis plane. On following this principle, you realise the symmetric of each point of the "J" highlighted by the isometric paper, and then - by tracing a line between such points - you actually obtain a "J" as in the picture. This error moves from a very deep misconception: the assimilation of a symmetry in the space as to a vertical plane to a plane axial symmetry.

The second problem in (b) is much easier both because of the experience gained with the previous problems and because the image to be represented is less complex than the previous one, despite the complexity of the object and the fact that a hidden part of it must be represented. The only difficulty, which to tell the truth we had not been foreseen, was the position of Alice's sticker: a teacher claimed that "Alice is outside the mirror" and therefore didn't represent her. Other errors concern some other forms of confusion between object and picture (the ranging between correspondent points hasn't been followed) and the wrong height of the object. Besides, nobody of the group ever considered that the object could have different configurations on ist back which were hidden in the given representation.

Among the problems in (c), the most interesting - as far as the production are concerned - is the second one. This problem asks to fill in with the missing lines the representation of an object of which the outline is given; it is also said explicitly that in the last two cases more than one object are likely to appear. The difficulty of the exercise is linked with the ability to imagine one or more objects, even in non-standard positions; at the only condition that its representation fit in the given outline. Not all teachers managed to solve this problem; some of them even declared that they couldn't see any suitable object for the last shape, whereas other teachers said they could see only one. The problem contained in (d) has been the most difficult: many didn't solve it and some even thought it to be definitely unsuitable for 11-year-old pupils; since it makes explicit reference to the cardinal points, that in the teacher's opinion the pupils can't manage properly in their real meaning, and because of the further difficulty of co-ordinating the three different representations towards its solution.

Here we cannot report some protocols and written comments by the teachers but these will be showed in the presentation.

Conclusive considerations

During the seminar the teacher's attitude has been gradually changing. At the beginning they were rather perplexed about the actual possibilities of introducing such activities in their programming, but they faced these problems themselves and made an accurate check-up of the difficulties and potentials they contain, which gave them new awareness and induced them to test such problems - yet presented in a naive way - in class, even if they weren't strictly framed in the programming. The problems were tested with 1st, 2nd and 3rd grade classes. As a matter of fact, we detected the teachers' difficulties in their performances too. However, let alone the pupils' performances on each problem, we would like to underline: a) higher flexibility than teachers on facing the problems, and in particular better answering to problems implying inquiry. b) remarkable pupils' liking and involvement in such kind of activity; c)identification and appreciation of the meaning of the activity.

The teachers, on the other hand, expressed their need to continue their study on these problems in order to create a three-year itinerary in which this activity could be properly framed by paying attention to the peculiarity of our syllabuses and goals in the teaching. Some interesting suggestions such as those devoted to the study of the representations of an object on horizontal and vertical plane from the fundamental viewpoints (north, south, east, west) were actually rejected, because considered more suitable to the lessons of the subject called "technical education".

The teachers' different opinions on this point are focused onto the need of clearing up about the themes and ways that should be at the basis of the teaching of geometry at middle school level, taking into account the need to conform its teaching in Europe.

Nonetheless, this study reveals how important it is for teachers to get operative training, which is rarely achieved through the theoretical study, which is the main component of our university courses and underlines the question of the study typologies in teacher training (at least in our country).

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From drawing to construction: teacher's mediation within the Cabri environment

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Abstract. Referring to a long term experimental project focused on the introduction to mathematical proof, this paper presents the analysis of a collective discussion taking place in a 9th grade class. The discussion deals with different strategies for constructing a square in the Cabri environment. The analysis has two objectives. On the one hand, to show the evolution of the justification process centred on the shift from checking on the product to checking on the procedure. On the other hand, to show how the relation to drawing is modified by the mediation of the Cabri environment as it is accomplished by the teacher.

1. Introduction

A long term experiment concerning geometrical constructions in two environments ("paper and pencil" and "Cabri") was carried out in the last years focused on the problem of introducing students to mathematical theorems (Mariotti, 1997, Mariotti et al. 1997). Geometrical constructions are elements of a field of experience. The term 'field of experience' is used after Boero & al. (1995) to mean 'the system of three evolutive components (external context; student internal context; teacher internal context), referred to a sector of human culture which teacher and students can recognise and consider as unitary and homogeneous'. In this report we shall limit ourselves to analyse a critical case of classroom interaction in order to discuss some elements of the external context and their relation to the internal context.

The evolution of the field of experience is realised over time through the social practices of the classroom. In this experiment, classroom verbal interaction is realised by means of '*mathematical discussion, i. e. a polyphony of articulated voices on a mathematical object, that is one of the objects - motives of the teaching - learning activity*' (Bartolini Bussi, in press). The polyphony of voices in this case concerns the dialogue between the voice of practice and the voice of theory about graphical construction. On the one side, the concrete production of a drawing on a sheet of paper is a practical activity, whose correctness is definitely controlled by empirical verification. On the other side, geometrical constructions have a theoretical meaning that overcomes the apparent practical objective. Every geometrical construction is actually based on a theorem (some theorems) that guarantees the theoretical control of the procedure by means of which it has been realised. As history witnesses (Heath, 1956, pp. 124-31), the relationship between a geometrical construction and the theorem which validates it, is very complex and difficult to be grasped by students: the theoretical control is not spontaneously achieved, but can result from the activities that the pupils perform within the chosen field of experience (Mariotti, 1996).The main motive of classroom activities (Bartolini Bussi, op. cit.) proposed to the students is the idea of geometrical construction at a theoretical level. Through the dialogue between the voice of the practice and the voice of the theory, the ability in creating and reading configurations, that is continuously practised in the production of drawings in both environments has to be enriched with the theoretical control.

The didactic problem concerns the ways of realising the evolution of the idea of construction: The key point of a theoretical approach to construction is that validation does not concern the product of a procedure (the concrete drawing), but the procedure itself. According to our experience (Mariotti, 1996) this change of focus is hard to achieve. The nature of the particular environment may foster this shift, providing a context within which the request about the procedure becomes meaningful, nevertheless the context itself is not sufficient and the intervention of the teacher becomes crucial. The following analysis of a collective discussion aims at showing the complexity of the process as well as pointing out the main elements contributing to its development.

2. The External Context

The external context is determined by the concrete objects of the activity (paper and pencil; the computer with the commands of the Cabri software; signs - e. g. gestures, figures, texts, dialogues). In this report we shall focus on the functioning of particular objects offered by the Cabri environment in relation to the internal context:

- the primitive commands and macros force to make explicit geometrical properties hidden in free hand drawing;
- the dragging function starts as a control tool to check the correctness of the construction, then becomes the external sign of the theoretical control.

In the concrete realisation of classroom activity, both elements may be used by the teacher as instruments of semiotic mediation (Vygotskij, 1978) .

In the Cabri environment (Laborde,1993; Laborde&Capponi,1994) the construction activity, i. e. drawing figures through the available commands on the menu, is integrated with the dragging function; as a consequence of dragging, each figure produced on the screen may or may not maintain its characteristic features, according to the procedure through which it was obtained. A natural problem arises:

why do some construction pass the dragging test and not others?

The dragging function can easily be accepted as a validating tool, but the problem must be shifted from validating by dragging, to explaining the 'proof by dragging' itself. In other terms, the need of explaining why a drawing is correct and/or foreseeing that it will be, must lead to consider the procedure through which it was obtained and check its correctness.

3. The teacher's role.

The role played by the teacher is fundamental in every mathematical discussion. In this case, the discussion is developed in a special context, Cabri constructions. Consequently in addition to the standard strategies that are used by the teacher to manage discussions in a whichever context (see Bartolini Bussi in press), we have strategies that are specific for the Cabri environment (microworld). We are especially interested in discussing two of these strategies that are related to a typical facility offered by Cabri, i. e. the "history" in the Miscellaneous Menu. It allows to reconstruct a figure step by step. The objects are redrawn successively, in a logical order. When a given construction is focused, the teacher shows the history in the "master computer" and recurs to two games:

1) the interpretation game, lead by questioning which could have been the intention or the goal of the author in making such construction; for instance the teacher can ask: why did the authors choose this operation? what is the use for?

2) the prediction game, lead by questioning which could have been the following step in this construction; for instance the teacher can ask how would you go on from this point?

Both games are possible because in the computer there a decontextualised, depersonalised and detemporalised copy of the construction is available. It can be shown without funnelling students by explicit comments or implicit information (gestures, and so on) towards the expected answer.

In the following sections we will analyse the critical starting point of activity on Cabri-figures, to show how the teacher negotiates the meaning of geometrical construction during mathematical discussions.

4. The scenario of the activity

The episode that we are going to discuss involved one of the experimental classes of the project (9th grade in a scientific high school (Liceo Scientifico)); 19 out of the 23 pupils in the class participated to this activity. This is the first activity and constitutes the very start of the long term experimentation.

The first part of the activity took place in the Computer room, where pupils sat in pairs at the machine. Pupils had a general expertise of the computer, but they never used Cabri; after a short acquaintance with the Cabri environment - they were let to freely explore the software for about half an hour - the following task was presented.

Construct a segment on the screen. Construct a square which has the segment as one of its sides.

As the teacher explains, pupils are asked to realise a figure on the screen and to write down a description of both the procedure and their reasoning. At this point, the term *construction* is ambiguous, but this is done on purpose: different interpretations of the task are expected, providing the basis for the following

discussion. Actually, the protocols collected contain solutions differently obtained referring to geometrical properties and/or referring to perceptual control: when the dragging function will be used these solutions will be differently transformed.
The teacher opens the discussion suggesting to analyse the solution given by Group 1 (Giovanni&Fabio). The solution was obtained drawing four consecutive segments, arranged in a square using perceptual control.

4.1 Is the drawing correct?

When the teacher asks pupils to judge this solution, pupils show to share a common objective. Although some of the solutions resorted to general interpretations, by using commands explicitly referring to geometrical properties, when the question about correctness is put pupils agree that the control must be exerted on the particular drawing; according to the well known definition of square, pupils suggest to measure the sides and the angles. The main elements, arising from the discussion, are the use of measure and the precision related to it.
With the aim of shifting the focus from the particular to the general the teacher puts the following question.

> 9 I. : I'd like to know if, in your opinion, this **always results** (enphasis on) in a square

The teacher aims to direct pupils towards a process of generalisation (**always results**), which should make the judgement move from the particular drawing to the class of figures related to it by the procedure used.
After few utterances, it is possible to observe a shift of attention and geometrical properties characterising a square are considered.

> 21 Paolo: One should see if the angles are equal.
> 22 I: Can you see it here?
> 23 Chorus: No, we cannot, let us measure the angles.
> ...
> 29 I: Is it sufficient ...
> 30 Coro: Yes, Yes it is ... All of them, ...
> 31 Marco: No, two of them are enough.
> 32 I: Which ones?
> 33 Michele & Francesco (others too): The opposite (angles)
> 34 Marco: No, all of them.
> 35 I: Let's find an agreement
> ...
> 63 Paolo: If a square has all its sides ... equal then it is sufficient to measure only two of the two opposite angles, if on the contrary the quadrilateral has all its sides different, one should measure three angles.
> ...
> 68 I: You all are convinced that one angle is sufficient ... also those who said two opposite (angles)? Giuseppe?
> 69 Giuseppe: Yes, one is enough.
> ...

The intervention of Marco (31) focuses on the possibility of reducing the number of measurements and the discussion begins, lasting for a good while, about the number of angles which is necessary to measure in order to check the figure.

The summary of Paolo (63) and the other interventions show that an agreement is found, but it still concerns the control of the drawn square.

4.2 Is the procedure correct?

The discussion is interrupted and resumed the day after. One of the pupils summarises the main points of the previous discussion, then the teacher presents the figure proposed by Group 1 (Giovanni & Fabio) and drags it.

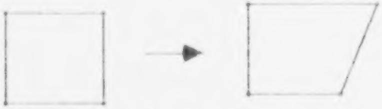

fig. 1 The construction does not stand the dragging test

12 I: What do you think of this figure
13 Chorus: that is a quadrilateral ... with different sides
14 Marco: with four different sides ...
15 I: Is it a square??
16 Coro No, No no

The solution was obtained using perceptive adaptation, thus after the dragging the figure is deformed; the teacher asks the pupils' opinion and immediately proposes another solution (Group 3 Dario& Mario).

17 I: O.K. Let's consider another solution ... I'll show you what has been done by Dario e Mario ... if I'll succeed
[...]
21 I: Well, I'd like to know your opinion about the construction of Dario & Mario
22 Marco: They did a circle then two perpendicular lines ...
23 I: Do you know from what did they start?
24 Michele: We can use the command "history".
25 I: Let's do it . They took a segment, then they ...
[...*the construction step by step follows*]
They drew a line perpendicular to the segment , then the circle ... in your opinion, what is it for ? What the use of it?
SILENCE
Is there a logic in doing so, or did they do it just because they felt like to draw a perpendicular line ... a circle ... Alex, tell us ...
26 Alex: the measure of the segment is equal to the measure reported by the circle on the perpendicular line.
27 I: You mean that the circle is used to assure two equal consecutive segments, the first one and that on the perpendicular line ... and the perpendicular ...
28 Chorus: is used ... to obtain ... an angle of 90°
29 I: I know that the square has an angle of 90° and four equal sides or three equal angles ... then let's see if it is true ... let's go on. Intersection between line and circle. They (Dario e

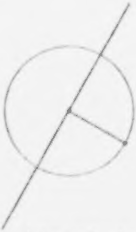

fig. 2 The first steps of the construction Group 3 Dario & Mario

Mario) determined the intersection point between the line and the circle ... why did they need that point?
30 Chiara: the intersection point between the line and the segment ..
31 I: and what should you draw from there ?
32 Chiara: a segment, perpendicular to the line
33 I: what else??
34 Chorus: parallel to the segment...
35 I: let's see what did they do ..

A first reaction comes from Marco, who starts to describe what he supposes that Dario & Mario did and suggests to use the command "history". Following this suggestion the teacher puts in execution the command, at the same time describing what has been done. At a certain point, she interrupts the description and starts the '**interpretation game**' (23): she asks the pupils to reflect and try to detect the 'motivations' for those actions. This game aims to provoke the first *shift from the procedure to a justification of the procedure itself*

The pupils are confused: in fact, the teacher's question is followed by silence. That shows the difficulty and the artificiality of the move leading from action to expliciting the motivation of this action.

But the teacher presses the pupils to find a "logic" in the procedure described (25). The following interchange is very interesting, revealing the functioning of the discussion in respect to the "motive" of shifting the control from the description of the procedure to the motivation of the procedure.

Alex (26) expresses the relationship between two of the segments according to the series of commands previously executed.

The teacher (27) reformulates the statement of Alex in terms of motivations "You mean that the circle **is used for** assuring two equal consecutive segments ...". The Chorus appropriates the terms used by the teacher and continues in terms of motivation.

The discussion continues developing the analysis started off by the teacher: going through the construction, the pupils are now asked to foresee the next step, motivate it and then compare it with the step recorded in the history. That is what we have called the **prediction game**.

In so doing, the pupils understand that properties, stated through geometrical relationships, guarantee that the final product **will always result** a square; at this point, it is possible to negotiate the acceptance of a Cabri figure as the correct solution of a construction task.

53 I: Can I say that the other drawing is a square?
54 Chorus: No
55 I: Before moving it was a square ... but, it does not last ... then, in your opinion, when asked to construct a square, what is better, a square which is always coherent, also when it is dragged, or a square which can become whatever else?
56 Chorus : A square which always remains a square
57 I: Well, in your opinion, what is the difference between the two constructions

The discussion continues: different solutions are compared. Mario suggests the idea of "a link among all the sides" and successively Fabio comes back to the term construction, which now assumes the precise meaning of the relationship among the geometrical elements involved.

When the chorus resume the term *construction* (120), the teacher, immediately institutionalises it:

> 121 I: Do you remember what I asked you at the beginning "what is a construction?" ... in this case we made a construction .. that means, ...

A final evaluation of the different types of solution is attempted and the term "construction" appears again: now its meaning has evolved.

> 181 Daniele: the 'first' (best) is that (Group 3 Dario & Mario)
> 182 I: Why is it so? What is the motivation?
> 183 Chorus: Because it is impossible to mess it up.
> 184 I: Why is it impossible to deform it? How was it constructed? What did they used?
> 185 Chorus: all in function ...
> 186 Fabio: They used a geometrical construction.

A construction is a procedure resulting in a figure which will not be deformed by dragging, the elements are related by characteristic geometrical properties; from now on the pupils agree on the acceptance of a solution in terms of the dragging test, but it is also clear that it is possible to explain why a solution is acceptable.

5. Conclusions

The first observation concerns the amount of interventions of the teacher during the discussion: she seems reluctant to leave the control to the class. This is certainly one of the features characterising a first approach to social interaction in a classroom which must be overcome; actually in this case, for both the teacher and the class this is the first experience of a collective discussion.

As far as the problem of construction is concerned, the difficulties shown in this short analysis will not disappear in a while, but there is evidence that the particular meaning of the construction act - and of its justification - emerged, so that it will be possible to come back to this meaning as long as it is necessary.

As expected, although not simple and spontaneous, the shift from drawing as a product to drawing as a procedure occurred.

According to our basic hypothesis the relation to drawing is modified by the mediation of the Cabri environment as accomplished by the teacher: our results confirm that the specificity of the Cabri environment is determinant in order to make sense of geometrical constructions highlighting specific aspects of them.

Besides the role of the dragging function which mediates the generality of a figure, the previous analysis reveals the importance of the "history" command as a mediation tool available to the teacher to trigger the **interpretation** and the **prediction games.** Those games represent a good example of mediation accomplished by the teacher: keeping the control, she leaves it to the pupils to make explicit the required operations and their motivations. It is important to remark that the "history" command provides the basis for the analysis, but it is not sufficient to accomplish the shift from operations to intentions: the software only shows the sequence of the steps, whilst the interpretation game introduces the point of view of motivation, which is reinforced by the prediction game. Both the games are based on the facilities offered by the software; the hystory command allows the reconstruction "step by step" of a figure, i.e. provides the sequence of the operations (decontestualised and depersonalised) through which the action with the goal that motivated it is to be reconstructed. Thus the software provides the teacher with the mediation tool of the history command to introduce the pupils into the games of motivations.

It is not a surprise that the interpretation and the prediction games are so effective. Similar games are used by the teacher in the voices and echo games described by Boero & al. 1997, where students are asked to interpret an historical source and to predict which solution could have been produced by the same author in a given situation. In order to start the process what is needed is a 'text' (the source in the case of the voices and the echoes game; in the case of Cabri constructions the sequence of operation reproduced by the history command) that can be analysed by students in a detached way, in order to live the author's part and to guess the author's intentions. It is the very presence of the software that transforms a personal construction into a depersonalised logical sequence of instructions that can be looked at by the author himself/herself in a detached way.

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"SHE SAYS WE'VE GOT TO THINK BACK": EFFECTIVE FOLDING BACK FOR GROWTH IN UNDERSTANDING.

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Abstract

Folding back is one of the key characteristics of the Pirie-Kieren theory for the growth of mathematical understanding. This paper illustrates ways in which folding back can be either an effective or an ineffective action for facilitating the growth of mathematical understanding.[1]*It highlights what an act of folding back must comprise if it is to lead to an enriched understanding for the learner.*

The notion of 'folding back'

The research on which this paper draws is based in the Pirie-Kieren model for the dynamical growth of mathematical understanding. The Pirie-Kieren theory has itself been fully presented and discussed in a number of previous PME meetings[2] and many of its features elaborated on, there and elsewhere.[3] The model developed to represent the growth of understanding contains eight potential layers of informal and formal mathematical understanding actions for a specific person and of a specified topic. These layers are named primitive knowing, image making, image having, property noticing, formalising, observing, structuring and inventising.[4] A diagrammatic representation of the model is provided by eight nested circles; each layer contains all previous layers and is included in all subsequent layers. Growth is seen to be the result of a continual movement back and forth through the layers of knowing, as individuals reflect on and reconstruct their current knowledge.

A key feature of this theory is the idea that a person functioning at an outer level of understanding, when faced with a problem at any level, that is not immediately solvable, needs to return to an inner layer of understanding. Such an invocative[5] shift is termed 'folding back'. The intention is then to extend current inadequate and incomplete understanding by reflecting on some inner layer activity, reorganising earlier constructs, or even generating and creating new images, should existing constructs be insufficient to solve the problem. The inner-level activity is not identical to that originally performed by the students, however. When folding back, students possess a degree of self-awareness about their understanding, informed by their operations at the outer level and so they are effectively building a 'thicker' understanding at the inner level, to support and extend their understanding at the outer level to which they subsequently return. It is the purpose of this paper to focus on the ways in which folding back facilitates the growth of mathematical understanding. In particular the paper will illustrate the way in which folding back can be either 'effective' or 'ineffective' for the learner. Merely engaging in an act of folding back is not a guarantee of greater understanding.

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[1] The paper stems from the research of Lyndon Martin which forms part of a DPhil thesis presently being submitted to the University of Oxford under the supervision of Professor Susan Pirie.

[2] For example, Kieren and Pirie, 1992, Kieren, Reid and Pirie, 1994, Martin, Pirie and Kieren, 1996.

[3] For example, Pirie and Kieren, 1992, Pirie and Kieren, 1994.

[4] For a more complete description of the model see Pirie & Kieren, 1991.

[5] Invocative here is used in the context of the model to describe a cognitive shift to an inner level of understanding. An invocative intervention is one which promotes such a shift.

Effective folding back:

Folding back occurs with a purpose, namely extend one's existing understanding which has proved to be inadequate for handling a newly encountered problem. It is in response to an obstacle that the learner re-visits earlier understandings, aiming to modify, collect, or build anew conceptions which will allow the difficulty to be overcome through an extended understanding of the topic.[6] Hence the inner layer activity is informed by what the learner already knows and by what they need to be able to do. For a learner who is able to use this extended understanding to overcome the original obstacle we term such folding back 'effective'.

it is important to note that this does not imply that the learner now has a complete solution. Further folding back may be required before a sufficiently extended understanding exists. The key feature is that the learner is able to return to the outer layer and apply newly extended understanding to the original problem . Continued working may yield another new and different obstacle for the learner, necessitating further back and forth movement, but this is distinctly different from being unable to make use of the new constructs at the outer layer. In the former case the understanding of the learner is still growing through a continual back and forth movement whilst in the latter it has been temporarily halted and is termed 'ineffective.'

Consider the following extract of two students working. Their problem is to determine the area of a segment of a circle. Having found the area of the sector they are now working on finding the area of the enclosed triangle:

Example One

Teacher	Tell me what you think you're doing here to find that shaded bit, what are you really doing?
Kerry	Trying to find out the area of the triangle...
Rosemary	Yeah
Teacher	Trying to find the area of the triangle.
Kerry	That's what we were trying to do but we can't do it.
Teacher	All right, what do you...Pop out that triangle and draw it for me there. Write on everything you know about that triangle...

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[6] For more detail on the different forms of folding back see Martin, Pirie and Kieren, 1996

Kerry *(drawing)*...it's six. That's six centimetres...It's thirty degree angle is thirty and that's six...

Teacher Good. So we know quite a bit about the triangle. How can we find the area of a triangle? What is its formula?

Kerry Base times height...

Teacher Nearly...something times base times height....half base times height.

Rosemary Oh yes..

Teacher Okay?

Rosemary ...or we do just base times height then half it...*(Pause as the look to the teacher for more guidance)*

Teacher Now what of those pieces of information do we know? Do we know the base? *(She points to the triangle the students have drawn and are trying to find the area of)*.

Rosemary Yep...

Kerry No...

Rosemary Yes...six

Kerry No, it's not six...

Rosemary Why's it not six?

Kerry Because the radius is six, that's coming across there.

Rosemary But surely if you take it from the middle point to wherever its going to still be six?

Teacher It depends what you call the base doesn't it? If you want to call this, (*Pointing to the chord*) this the base then no you don't know it...

Rosemary No, well I'm calling either of these lines the base...*(Pointing to the radii)*

Teacher You're calling that the base...

Rosemary In which case it would be six...

Teacher So supposing you do know the base is six.

Rosemary Six.

Although Rosemary and Kerry know that they need to find the area of the triangle they have not been able to do this, within the present context. The teacher suggests that they think about it separately from the question. Through her intervention she is suggesting that the pupils fold back to their primitive knowing and collect from this inner layer the formula for the area of a triangle. The teacher is confident that the students know this formula and she does not therefore try to teach them anything about finding the area of a triangle. Here, it is appropriate for them to re-call or to re-collect an earlier understanding and image. They are seen, with help from the teacher, to collect the correct formula, and Rosemary demonstrates her ability to state the rule in her own words. With the question "Now what of those pieces of information do we know? Do we know the base?" the teacher prompts the pupils to return to their outer levels of thinking and to make and use an image for the area of the segment of a circle. They now need to consider the triangle as related to a part of the circle. Kerry and Rosemary clearly move as intended, they talk about the 'radius' and 'middle point' and relate their measurements on the triangle to the circle. They are now image making again, informed by their existing images and understandings prior to folding back.

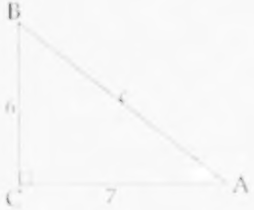
together with their knowledge of the triangle collected from their primitive knowing. Here it was the intervention of the teacher that enabled this successful return.

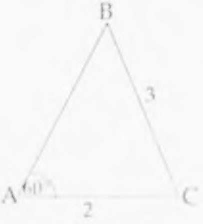
Kerry and Rosemary engaged in an act of collecting, which was appropriate, since the mathematics they collected was relevant and correct. They were able to return to the original problem, and continue image making within the context of the circle. We see that, by the end of the episode, the folding back activity proved to be effective, although if we examine the dialogue carefully we notice that, following the return to the original problem, Kerry did not initially see her collected knowledge as useful; she did not realise that they knew the necessary dimensions of the triangle.

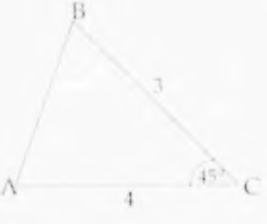
Ineffective folding back:

In contrast to the above extract consider two further classroom examples. The first is from a lesson in which two boys, Tim and Donald, are doing an exercise that requires them to find the missing values in various triangles, the first three of which are right-angled triangles.

Example Two







Tim Tan A is opposite over adjacent, so six over seven, Donald (*using calculator*) point eight five seven, root, no, tan is (*pause*) what? nought point nought one four etc.? what?

Tim Inverse tan

Donald Oh, yeah! Inverse, tan, point eight five seven, inverse, tan, is forty point six

Tim Tan B , seven over six

Donald Seven, divide, six, inverse, tan, forty-nine point three.

Tim And c squared is six squared plus seven squared ...

Donald Six squared plus seven squared, square root, nine point two one.

Tim Next one. Oh, how do we know which is the right angle? We can't! (*Tim gets up and goes away to talk to the teacher. He returns*)

Tim She says we've got to think back a few weeks.

Donald Oh! You, we need those rules. You know. The ones with sin something and cos.

Tim But you still need the right angle, so's you don't know which is the hypotenuse

Donald No, no there's another one, erm, it's, it's, oh come on, its,

Tim Sin A is opposite over hypotenuse

Donald No, I've got it. a sin A equals b sin B equals c sin C. That's it. So three sin sixty equals two sin B, so ...(*uses calculator and calculates the value for angle B, using their incorrect formula*)

(*....... several similar examples later*)

Tim Boring (*yawns and stretches*) four sin forty-five equals (*pause*) what? four is b, but you don't know B and three is a but you don't know A. Is it? Look Donald, this picture. It's got to be wrong. Drawing's wrong. (*Puts up his hand*) Mrs. Smith.

Donald No it's the cos one.' a b c cos C', so three times four times (*pause*) we don't have c, oh I don't know! (*calling out*) Mrs. Smith!

Teacher So what's the problem?

The teacher, having ascertained the boys' difficulty, did not then simply offer the correct formulae, but spent some time taking them back through their earlier work deriving the Sine and Cosine Rules, saying at one point "you wont remember it if you don't understand it".

We see Tim and Donald trying to work at a formalising level, applying 'known' formulae to solve for the dimensions of given triangles. In a fashion similar to Kerry and Rosemary, they call upon the teacher's help when they meet an obstacle to continued working, and with her prompting, fold back to collect the sine rule that they have previously learned. Unfortunately they re-collect an incorrect version of the rule, return and apply it, believing that they have successfully completed the original problem. Later the two boys encounter another obstacle and, probably as a result of the teacher's earlier invocative intervention, Donald recalls another formula - again incorrectly. This time, however, they are unable to make use of their erroneous re-collected knowledge. Although folding back was an appropriate cognitive act, it did not enable the students to continue working on the original problem. Rather than just enable the boys collect the correct formulae, however, the teacher then got the boys to fold right back to their earlier learning and spend some time having them work on their images for the two rules and their applicability in certain situations. To be effective here, the folding back needed to incorporate an element of working at an earlier level, before returning to the originating problem.

Example Three

Here we see three students starting to solving a problem together.

Jen (*reading the question*) "Hare challenges Tortoise to a 1 mile race. Hare's average speed is x mph and Tortoise's is y mph.. After half a mile Hare stops and plays for 5 hours" oh god! "and runs the last half mile to arrive just behind Tortoise. What are their relative speeds?" Relative speeds. OK, This is just one of those train problems. You know, relative speeds
Sally Like when they started at different stations and pass each other.
Mark So is it the one where you add the speeds or take them away?
Jen You add them if it's like one of them stops and the other goes twice as fast.
Sally If one's going at 50 mph and the other is doing 60 then they collide at 110. But if they are just passing each other it's 50 minus 60. You know 10 miles relative speed.
Jen the hare stops so it's like the tortoise is going much faster, right?
Mark No. Look they're going the same way so it's a take away. x minus y.
Jen But the hare doesn't pass the tortoise.

Jen, Sally and Mark immediately fold back to a set of problems - trains - that they see as similar to the new problem, The words "relative speeds" have acted as an invocative intervention for them. they all fold back, although Mark seems to be trying to collect a formula while Jen and Sally seem to be working with their images for the class of 'train' problems in order to understand the Hare and Tortoise question. When they return to the new problem, however, the folding back appears to have been ineffective, as none of them is yet able to solve the problem. Here collecting was an inappropriate action for Mark, because the problem does not allow solution by simple application of a learned routine. For Jen and Sally, folding back to work with their images was appropriate, but the actual image making at this point is insufficiently focused to be of value

Conclusions

Although a learner may well return to an inner level of activity, this in itself is not sufficient to guarantee that the learner is able to make sense of the initial 'incoherent or incomprehensible' situation. In some cases, folding back can still leave the learner unable to make progress and extend their understanding of a concept or situation. Although Jen, Sally and Mark spontaneously fold back this does not become an effective act for them.

In contrast, Rosemary and Kerry fold back to try and collect the formula for the area of a triangle, this is appropriate as they have met the formula previously. The act of collection is itself successful; they are able to collect the required formula. They are then able to use the formula to advantage; the folding back was effective. If, however, they had collected the wrong formula, perhaps that for the area of a circle then the folding back would have become ineffective.

For the understanding of a learner to continue to grow it is necessary for the learner to fold back in an appropriate way to facilitate growth and to be able then to use this inner layer activity effectively to solve the original problem at the outer layer. There seem to be three key aspects of folding back which contribute to the resulting effectiveness of the act. Firstly learners need to be aware of what the limitations of their present understandings are, in terms of the mathematics, and also aware of what kind of inner layer activity they must undertake once they have folded back. If, for example, learners fold back to collect when they actually need to work at an inner level, extending understanding, then the action is likely to be ineffective. In example three it is clear that whilst Mark is able to fold back and is aware that his present understanding is insufficient to allow him to solve the problem, he does not know precisely enough what it is he does not understand. He believes strongly that he knows what he needs to know and that if he can re-member and collect this piece of mathematics he will be able to use it. The reason his folding back is nonetheless ineffective is that his understanding of the mathematics which he is trying to collect is insufficient. Here it is the act of folding back to collect which is inappropriate. Instead, there is a need for him to engage in a different form of folding back, to work with Sally and Jen extending their images, until they are in a usable form.

The second aspect is whether the actual mathematical activity at the inner layer is appropriate. If, for example, collecting is the appropriate kind of inner layer action, it is then necessary for the learner to actually collect a useful piece of mathematics. This can be seen in the case of Tim and Donald, who, although they fold back and engage in collecting, do not access the piece of mathematics needed to allow the folding back to be effective. Whilst the act of collecting is appropriate, what they actually collect is not.

The third key aspect which determines whether folding back is effective for a learner is his or her ability to relate inner layer actions and new understandings to the original problem. It might be assumed that a learner who is folding back does so with purpose and that using the extended understanding in the initial problem would be somewhat automatic. For some students though, this barrier was not insignificant and being able to use their extended understanding proved as difficult as actually performing the inner layer actions. This is apparent in the continuation[7] of example three where Sally, who appears here to be recalling an appropriate area of mathematics, still cannot use it in the context of the original question, even after considerable further work with her peers on their images.

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[7] not included here for reasons of length of paper.

For some students further intervention, perhaps from a teacher, is needed to provoke or direct their application of their inner layer actions to the original problem, as is seen in example one. Research suggests that the need for an intervention often occurs where students have folded back and either moved out of the topic to work there or to collect something from their primitive knowing. In such cases, the learner is deliberately not thinking in the context of the initial problem, instead they are working on what is often a very different area of mathematics. For example, Kerry and Rosemary later work on trigonometry whilst solving a problem dealing with the area of the segment of a circle. It would seem that in such cases it is vital for the students to remember the original purpose that is being used to inform the inner layer actions. Where the learner appears to be moving away from this aim then the teacher has a role in monitoring the student activity and if needed in facilitating movement back to the problem through an appropriate provocative[8] intervention.

This paper has highlighted that to merely engage in the act of folding back is not necessarily enough to facilitate the growth of mathematical understanding. Whilst folding back is seen as a mechanism through which understanding can grow, there is clearly a need to pay close attention to the detail of the act of folding back itself. It is through folding back in an appropriate way, through engaging in appropriate associated mathematical actions and through being able to apply the new understanding to the original problem that students' growth of mathematical understanding can be engendered.

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[8] Provocative here is used in the context of the model to describe a cognitive shift to an outer level of understanding. A provocative intervention is one which promotes such a shift.

YOUNG CHILDREN'S BELIEFS ABOUT THE NATURE OF MATHEMATICS

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This paper reports part of a doctoral study of children's beliefs about the nature of mathematics, the nature of learning, and factors within learning environments that are perceived to impact upon the learning of mathematics. Eight children were interviewed on ten occasions. This report focuses on young children's beliefs about the nature of mathematics. Findings discussed draw on interview responses from two children. It is argued that it is important that teachers are aware of children's beliefs about the nature of mathematics, that it is possible to gain insights into these beliefs, that they can be subtle and complex, and that children may see mathematics more broadly than has been reported by some researchers.

Investigating children's beliefs about the nature of mathematics

Just as children learn mathematics in different ways and through negotiation of meaning construct their own understandings (Ernest, 1991), they also construct beliefs about the nature of mathematics. These may affect many aspects of their learning; perceptions of what mathematics is may influence approaches to problem solving in mathematics (Frank, 1988; Schoenfeld, 1987), may influence the nature of children's participation in meaningful mathematics learning (Franke & Carey, 1997), may impact upon conceptions of specific topics in mathematics, and may affect attitudes, performance, confidence, perceived usefulness of mathematics, and choice of courses or careers (Kouba & McDonald, 1987). Learners' perceptions of what mathematics is also influences interpretation of what is taught (Lindenskov, 1993), and consequently what is learned.

The investigation of children's beliefs can assist teachers to come to know, and hence to better cater for, individual learners. By knowing children's perceptions of the nature of mathematics, and thus having increased awareness of the perspectives children bring to their learning, teachers are better informed when interacting with individual children. Teachers can take children's beliefs into account when considering maintenance or refinement of elements within the mathematics classroom such as style of tasks, classroom organisation or the locus of decision making. Also, there is the potential for those beliefs which are less central (Rokeach, 1968) and those held with less conviction (Thompson, 1992), to be open to change or manipulation by outside influences such as the teacher, and thus for undesirable beliefs (Garafolo, 1989), to be addressed.

Reports of previous research indicate that for the majority of children, mathematics is an objective and certain body of knowledge. For instance, Garafolo (1989) refers to the widespread existence of students seeing mathematics as a collection of rote, mechanical procedures leading to correct answers, and of the teacher and textbook as authorities on mathematical truth. The emphasis on mathematics as a rule-governed activity appears present within students both at the primary and secondary levels (Cobb, 1985). Mathematics is seen also as "divorced from real life, from discovery and from problem solving" (Schoenfeld, 1987, p. 197); it is perceived as static, that is, as already having been created or discovered (Spangler, 1992).

Children tend to focus on the computational or number aspect as the essence of mathematics (Cotton, 1993; Frank, 1988; Kouba & McDonald, 1987; McDonald & Kouba, 1986; Spangler, 1992). For example, Kouba and McDonald (1987) and McDonald and Kouba (1986) found that for primary and junior secondary children the presence of explicit numbers and operations in a situation was a major factor in identifying the presence of mathematics. There is little evidence of children identifying other areas as mathematics. The narrowness of beliefs of primary school children, with geometry, statistics and probability generally not accepted as being within the domain of mathematics (McDonald & Kouba, 1986), contrasts with the range of content of the intended curriculum at the primary level (e.g., Australian Education Council, 1991; National Council of Teachers of Mathematics, 1989).

From studies such as those above, it appears that the possibility of complexity or subtlety of an individual's beliefs, or of beliefs differing between individuals in the same class, have received little attention in previous research; investigation of students' images of mathematics appears limited in some respects (Ernest, 1996). Thus there is potential for further indepth investigation of individual children's beliefs - a challenge taken up in the present research.

The research framework

The overall study investigated children's beliefs about the nature of mathematics, the nature of learning, and factors perceived to influence the learning of mathematics. For reasons discussed elsewhere (McDonough, in progress), the research did not subscribe to a particular model of developmental learning such as that of Piaget (Piaget & Inhelder, 1969), or a hierarchy of cognition such as that of van Hiele (Pegg, 1985), or a taxonomy of learning such as the "SOLO Taxonomy" (Biggs, 1991), but the framework for the research evolved within the research. It was developed according to the assumption that children's beliefs about learning and about mathematics, underpin, and are intertwined with, their perceptions of themselves as learners of mathematics. As Pajares (1992) noted "subject specific beliefs, such as beliefs about . . . mathematics . . . are the key to researchers' attempting to understand the intricacies of how children learn" (p. 308), suggesting an interaction between different facets of children's learning of mathematics, especially noting the importance of subject specific beliefs. The investigation of children's beliefs about the nature of mathematics began as a prerequisite to the investigation of beliefs about factors of influence in learning mathematics, but evolved as an informative and interesting element of the study in its own right.

Figure 1 presents a model illustrating the hypothesised relationship between the three areas of interest in the research. Children's beliefs about mathematics and about learning are represented as underpinning their beliefs about factors in the learning environment that impact upon their own learning of mathematics. In this cyclic model, developed within the research, the factors of influence, in turn, impact upon personal beliefs about the nature of mathematics and learning.

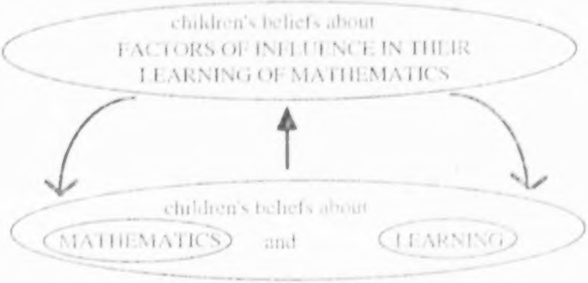

Figure 1. Children's beliefs: The basic model

The research methods

The eight research participants in the study were aged eight to nine years and came from two classes in two schools. A high achieving and low achieving male and female were chosen from each class by the class teacher.

Thirty procedures were developed or chosen for the research and deployed in ten one to one interviews with the researcher, each of 20 to 30 minutes duration, conducted over a period of five months. The procedures, suitable for use with children of eight to nine years, included drawing, describing, sorting words and pictures, building, and a small amount of writing. In regard to the nature of mathematics, children were given tasks including choosing, and explaining choice of, mathematical activities from photographs showing a range of school and non-school situations, developing a "personal dictionary" definition for mathematics, and completing the sentence "Maths is like ....", followed by discussion. Interviews were audio-taped, detailed field notes were written, and products were collected. The type and range of procedures allowed expression of subtleties and complexities in young children's beliefs that seem not to have been identified by previous research.

The analysis of data was not structured according to a pre-existing theory, but themes (van Manen, 1990) were allowed to emerge from the children's data

Themes commonly drew on responses to more than one interview task. Samples of themes that arose in relation to the nature of mathematics are "Mathematics as content or action", "Mathematics for an everyday or non-school purpose", "Comparisons related to number and measurement", and "Mathematics as answers", with themes differing for each child. Thematic analysis facilitated portrayal of beliefs reflecting children's individual constructions and orientations; this was compatible with the constructivist perspective that underpinned the research.

While acknowledging differing definitions of beliefs (e.g., Pajares, 1992), the definition of beliefs as given by Rokeach (1968) as "any simple proposition, conscious or unconscious, inferred from what a person says or does, capable of being preceded by the phrase, 'I believe that . . .'" (p. 113) provided the baseline for the approach taken in the present study. Beliefs may be cognitive in nature, such as in the statement "The sum of the angles in a triangle is 180 degrees", or may be affective, such as in the statement "Mathematics is boring". Beliefs about the nature of mathematics of the two children discussed in this report emerged mainly as relating to the content of mathematics, with consideration of what constitutes, and the purposes and location of, mathematical activity.

Findings

The full report of this research (McDonough, in progress) gives a detailed account of the beliefs of the two children discussed here including reference to interview excerpts from which portrayals of each child's beliefs were developed. To illustrate the possible complexity and subtlety of beliefs about mathematics held by children as young as eight to nine years of age, summarised data from the two children, of the pseudonyms Cara and Emily, are presented here. Schematic diagrams of the overall findings for each child are included, indicating the individuality of perceptions of the nature of mathematics, but with a focus for both mainly on the content of mathematics. The two children were chosen for this paper as they communicated beliefs that differ from those suggested by previous research. Cara showed complexity and breadth of belief about the nature of mathematics, with emphasis on measurement and estimation. Emily emphasised mathematics as number, suggested by research as a common perspective, but closer analysis revealed unexpected complexity and subtlety within her beliefs.

Cara

Cara, a grade three student chosen by her teacher as a low achieving female, began the interviews as an eight year old and turned nine during the data collection.

There appeared to be nuances and complexities within Cara's beliefs about mathematics. Although number appeared significant for Cara, at times related to the getting of answers, she did not, for example, see mathematics as relating mainly to number, as previous research has found is common for primary school children. Measuring and estimating in mathematics appeared salient and of personal relevance to Cara, with estimating related at times to guessing. This suggests Cara saw the processes of mathematics more broadly than calculating to gain correct answers. Cara appeared also to hold some degree of personal affinity with mathematics, particularly with the ideas of measurement.

Figure 2 is a schematic synopsis of Cara's beliefs about mathematics developed from her descriptions of mathematics and mathematical activities. This diagram indicates the complexity of Cara's beliefs, and the interrelatedness she portrayed. It shows also what appeared as two uncertainties for Cara (indicated by broken lines): firstly, whether or not to classify as measurement some activities included by the researcher as possible examples of informal measurement, and, secondly, whether estimation and guessing are the same. The inclusion of estimating and measuring as mathematical activities and the affinity with measuring as mathematical activity suggest a different picture of beliefs about the nature of mathematics from that portrayed in previous research findings.

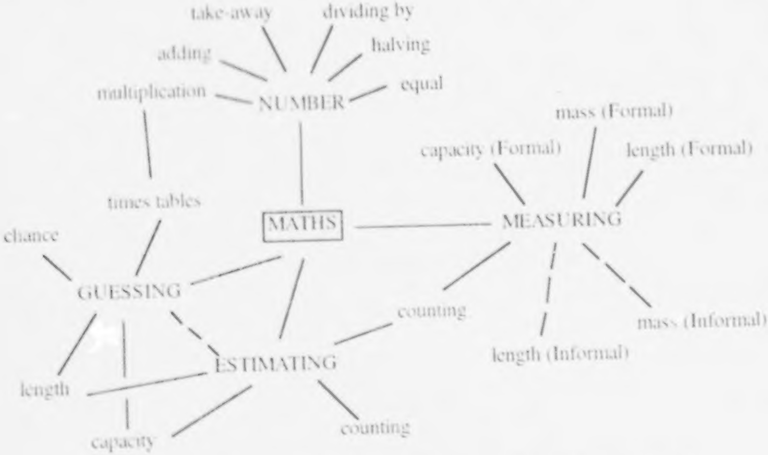
take-away
dividing by
adding
halving
multiplication
NUMBER
equal
mass (Formal)
capacity (Formal)
length (Formal)
times tables
MATHS
MEASURING
chance
GUESSING
counting
mass (Informal)
ESTIMATING
length (Informal)
length
counting
capacity

Figure 2. Cara's beliefs about mathematics - a schematic summary.

In regard to number, Cara seemed to be mostly product orientated, with the product obtained through calculation. She portrayed situations at school or in an environment where the activity had mainly a schooling purpose.

Measurement situations identified as mathematical included both school and non-school, with an emphasis on the use of formal measurement in non-school environments for those situations Cara proffered. While the product appeared an important element of the situations she described at home, Cara gave at least equal, if not more, emphasis to the process which often involved physical activity through measuring and sometimes cognitive activity through estimating. The situations in which informal measurement was a feature were at times not considered as measurement as such by Cara, but were considered mathematical, often because of their estimation component. These included both school and non-school activities.

Cara portrayed mathematics as multi-dimensional; she included activities in home and school environments and included both product and process dimensions.

Emily

Emily, a grade three student who attended a different school from Cara, was eight years of age throughout the data collection period. She was selected by her teacher as a high-achieving female.

The overall impression from Emily's data is an emphasis on mathematics as number and related operations, suggesting a simplicity of belief. However, her beliefs hold subtlety and complexity which become apparent from the analysis structured according to themes which emerged from the data. Figure 3 portrays the interweaving of concepts within Emily's beliefs. Emily's diagram is different in structure from that of Cara, due to different links, relationships and emphases in beliefs. Figure 3 provides one schematic portrayal of Emily's beliefs and makes clear, for example, her belief that measurement is not always mathematics, and that numbers are necessary for a situation to be mathematical although they are not always sufficient. The broken line for the "Informal capacity" entry represents conflicting responses by Emily to the situation of measuring rice for cooking. On one occasion, indecision was voiced regarding whether this situation is mathematical, while on another occasion the same situation was volunteered by Emily as mathematical. The italicised entries represent the number and measurement examples discussed but not identified as mathematics.

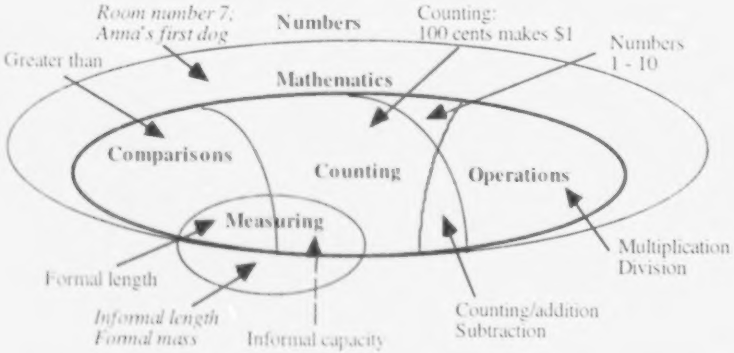
Room number 7;
Anna's first dog
Numbers
Counting:
100 cents makes $1
Numbers
1 - 10
Greater than
Mathematics
Comparisons
Counting
Operations
Measuring
Multiplication
Division
Formal length
Informal length
Formal mass
Informal capacity
Counting/addition
Subtraction

Figure 3. Emily's beliefs about mathematics - a schematic summary.

Emily in some ways held more narrow views of mathematics than Cara, but perhaps held more established views, with little uncertainty or contradiction apparent in her responses. Emily seemed to relate mathematics more closely to addition and counting, as well as to comparisons, multiplication, subtraction, division, and graphing, whereas estimating and measuring seemed to be the most salient aspects of mathematics for Cara. The paucity of reference to space, chance, and data as mathematics concepts by Emily and Cara is compatible with earlier findings.

Emily saw mathematics mainly as a school activity with some application for others in non-school environments. In contrast, Cara volunteered many mathematical activities from school and non-school environments both for herself and others, and for school and non-school purposes.

Concluding discussion

This report had three intentions. The first, to argue that children's beliefs about mathematics are important, was addressed by reference to many facets of mathematics learning which may be affected by a learner's beliefs about mathematics. The second, that researchers and teachers can gain insights into the beliefs of children of age eight to nine years, was supported by reference to findings from two children whose beliefs were investigated through the use of a range of interview procedures. The third intention, to demonstrate the possible complexity, subtlety and breadth of young children's beliefs about mathematics, was approached by presenting schematic and verbal synopses of beliefs about mathematics of two interviewees from the broader doctoral study.

The research framework included investigation of children's beliefs about the nature of mathematics as one element within the larger study. This report claims that study of children's beliefs about the nature of mathematics was an important and informative study in its own right. By investigating children's beliefs, the research adds to the understanding educators have of the possibility of breadth, complexity, and subtlety of beliefs held by children of only eight or nine years of age. The identification of different linkages, orientations and emphases in Emily's and Cara's beliefs about the nature of mathematics concurs with the constructivist orientation taken in this research.

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RECOGNITION OF ANGULAR SIMILARITIES BETWEEN FAMILIAR PHYSICAL SITUATIONS

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To investigate the relationship between concrete experience and the angle concept, a sample of 144 students in each of Grades 2, 4 and 6 were questioned about the similarities they saw between nine familiar angle situations. Dynamic similarities were only reported between moveable situations, whereas static similarities were reported between all situations. Static angular similarities were most often reported between situations, both fixed and moveable, where both arms of the angle were visible. The results support a refocussing of initial teaching about angle away from the turning aspect and towards the identification of the arms of an angle.

There has been an increasing trend in recent years to base elementary mathematics teaching on students' everyday experiences (Australian Educational Council, 1990; National Council of Teachers of Mathematics, 1989). The aim has been partly to make mathematics more interesting and meaningful to students and partly to strengthen their understanding (Hiebert & Carpenter, 1992).

In particular, several researchers have argued for, or attempted to use, concrete experience as a basis for teaching ideas about angles (Clements, Battista, Sarama, & Swaminathan, 1996; Lehrer, Fennema, Carpenter, & Ansell, 1993; Wright & Adams, 1992). Recent curriculum documents also recommend this approach. For example, the New South Wales curriculum guide (NSW Department of School Education, 1989) suggests for the first introduction to angle in Grade 3 or 4 that students make a concrete angle model consisting of two joined strips. Students are expected to see that the size of the angle is related to the amount of turning from one strip to the other, in accordance with the stated definition of angle as "the amount of turning between two lines about a common point" (p. 79). They then match their angle models to various "corners" and finally look for "everyday turns" inside and outside their classroom.

Characteristic of an experiential approach to concept development is the expectation that students will recognise that many situations which superficially appear to be different are actually similar. For example, the NSW syllabus assumes that students will recognise that there is a specific similarity between fixed and moveable angle situations, namely that they can both be represented by the concrete angle model.

Similarity recognition has been seen by some psychologists as the first stage of concept formation. For example, Skemp (1986) defined abstracting as "an activity by which we become aware of similarities ... among our experiences" and a concept as "some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class" (p. 21). The recognition that superficially different angle situations are similar is therefore opening. Static similarities referred to a common geometrical configuration. For example, students said "they both have two lines" or "you can put the tile in the corner of the junction". Responses such as "the scissors can be opened to show the corners of the tile" were also classified as static because (a) they refer to the resulting position of the moveable model and not to the movement which produces it, and (b) there is no implication that the fixed situation has any movement associated with it.

Task 2 The interviewer set one situation to show an angle of 45° (by moving a moveable model or selecting a fixed model). Students were then asked to set the other situation to "show the same as this".

This task was scored as correct if the correct fixed model was selected or a moveable model was turned between 30° and 60°.

Task 3 Students were asked to use a bent straw to show how they knew the two settings were the same.

This task was scored as correct if the straw was laid on both models in such a way as to clearly indicate both arms and the vertex of the implicit angle. Specific error limits were defined for each model; for example, a placement of the straw on the scissors was scored as correct if the arms were placed anywhere along the blades with the vertex in the region where the blades crossed.

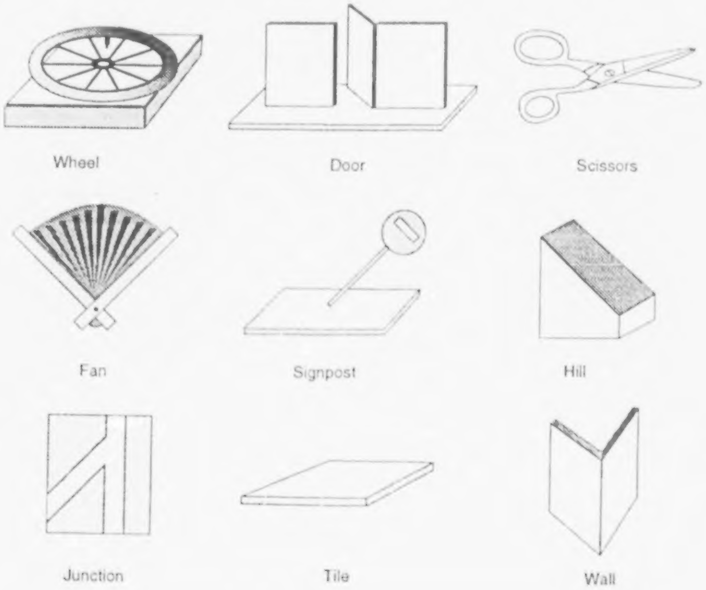

Figure 1: Models used to represent selected physical angle situations

fundamental to the development of the angle concept. The angle concept itself arises by abstracting the similarity so that it becomes a new mental object (Greeno, 1983).

There has been little research into the question: How well can young children recognise the similarity between different angle situations? There are many superficial differences between physical angle situations (Mitchelmore & White, 1997), and these differences may well obscure the underlying similarity (Mitchelmore, 1997, in press).

We recently completed a large-sample investigation of how well young students recognise angles in nine familiar, widely-varying physical angle situations. The purpose was partly to find what concrete examples might best be used in the initial teaching of angle concepts. We report on the part of this study which deals with the similarities that students recognise between different situations, in particular between fixed and moveable situations.

METHOD

Sample

The sample was gender-balanced and consisted of 144 children in each of Grades 2, 4, and 6 chosen from six schools in Sydney.

Situations

Nine physical angle situations were used: wheel, door, scissors, fan, signpost, hill, road junction, tile and walls. The first four of these are moveable while the last five are fixed. Each moveable situation was represented by a single model. Each of the fixed situations was represented by a set of three models representing a "neutral" configuration (angle 0° or 90°), an angle of 45°, and a "middle" angle (22.5° or 67.5°). Adjustable models of the fixed situations were deliberately not used, in order to avoid artificially suggesting movement. Examples of the models used are shown in Figure 1.

Twelve combinations of 3 situations were chosen in such a way that each of the 36 situation pairs occurred exactly once. Each combination was presented to 6 girls and 6 boys at each grade level, with order counterbalanced, in individual interviews conducted by a trained research assistant.

Tasks

In the interviews, the word "angle" was only used if students introduced it. The interviewer first showed students the three models, one at a time, and asked how the angle implicit in each situation could be represented by a bent straw. (The results of this part of the investigation are reported in White and Mitchelmore (1997).) She then presented the same models two at a time and gave students three tasks:

Task 1 Students were asked, Is there anything the same about these two situations? Neutral prompts were given until the student reported no further similarities.

Responses to Task 1 were classified as irrelevant, dynamic or static. Irrelevant similarities, such as "they are both made of wood", did not relate to angles. Dynamic similarities referred to the same movement in both situations, such as turning or

RESULTS

Tables 1 to 3 summarise the responses to Tasks 1 to 3 respectively. These tables refer to averages taken over 6 moveable-moveable situation pairs (N=72 responses at each grade level), 20 fixed-moveable pairs (N=240) and 10 fixed-fixed pairs (N=120). Variation between individual pairs of situations will be discussed separately below.

Table 1

Percentage of Task 1 responses indicating dynamic and static similarities

Situation pair	Grade 2	Grade 4	Grade 6
	Dynamic similarities		
Moveable-moveable	64	60	68
Fixed-moveable	2	0	0
Fixed-fixed	0	0	0
	Static similarities		
Moveable-moveable	4	74	86
Fixed-moveable	32	63	79
Fixed-fixed	50	81	89

As Table 1 shows, students almost never reported a dynamic similarity between a fixed situation and another situation. The overall dynamic similarity recognition rate between moveable situations was fairly constant at about two-thirds.

By contrast, static similarities were readily recognised not only between fixed situations but (especially after Grade 2) also between fixed and moveable situations and even between moveable situations. Within each category of situation pair, the static similarity recognition rate increased monotonically from Grade 2 to Grade 6.

Table 2

Percentage of Task 2 responses showing correct angle size matching

Situation pair	Grade 2	Grade 4	Grade 6
Moveable-moveable	56	89	90
Fixed-moveable	43	75	89
Fixed-fixed	73	91	97

Tables 2 and 3 show that, in all three categories of situation pair in both Tasks 2 and 3, the percentage of correct responses increased steadily from Grade 2 to Grade 6. Also, at each grade level, the order of facility was almost always the same: Fixed-fixed was the easiest, then moveable-moveable, then fixed-moveable. It is notable that, with the exception of Grade 2 students responding to moveable-moveable pairs, the same patterns also occur in the static similarities in Table 1.

Table 3
Percentage of Task 3 responses indicating correct angle positions

Situation pair	Grade 2	Grade 4	Grade 6
Moveable-moveable	36	61	81
Fixed-moveable	28	68	80
Fixed-fixed	51	82	89

A similarity index
The similarities between the patterns in Tables 2 and 3 and the static similarities in Table 1 suggested that all three tasks might be measuring a common construct. To investigate this possibility further, three "similarity scores" were calculated for each of the 36 pairs of tasks at each grade level: the percentage of students reporting a static similarity in Task 1, and the percentages giving correct responses in Tasks 2 and 3. The correlations between these three scores in Grade 2 (0.61, 0.68 and 0.72), and the correlations between the first and the third scores in Grades 4 and 6 (0.82 and 0.83) supported the hypothesis of a common construct. (The remaining inter-correlations were somewhat smaller, apparently due to a ceiling effect in Task 2.)

Tasks 1 and 3 clearly involve the recognition of a common configuration, thus showing that the common construct is static similarity recognition. We infer, therefore, that Task 2 also measures static similarity recognition. For example, students may match the angle sizes on a wheel and a fan not by comparing the two movements but by comparing the resulting configurations.

The identification of a common construct across the three tasks justified averaging the three similarity scores to obtain a "static similarity index" for each pair of situations at each grade level. For most pairs, the static similarity index increased monotonically from Grade 2 to Grade 6; the average indices were 41%, 74% and 86% respectively.

Similarity between situations
Hierarchical cluster analysis of the static similarity indices indicated one main cluster of similar situations at each grade level:

- In Grade 2, there were three situations in the main cluster (walls, junction and tile) with scissors nearby.
- In Grade 4, the main cluster consisted of walls, junction, tile and scissors. There were also two secondary clusters: door and fan, and hill and signpost.
- In Grade 6, there was only one cluster consisting of walls, junction, tile, scissors, signpost and fan.

Multidimensional analysis confirmed the pattern of a core of similar situations gradually expanding to include more situations at successive grade levels. The data are well described using facet theory (Levy, 1985) by the five-partition modulating model shown in Figure 2[1].

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[1] The authors are indebted to David Cairns for drawing their attention to facet theory.

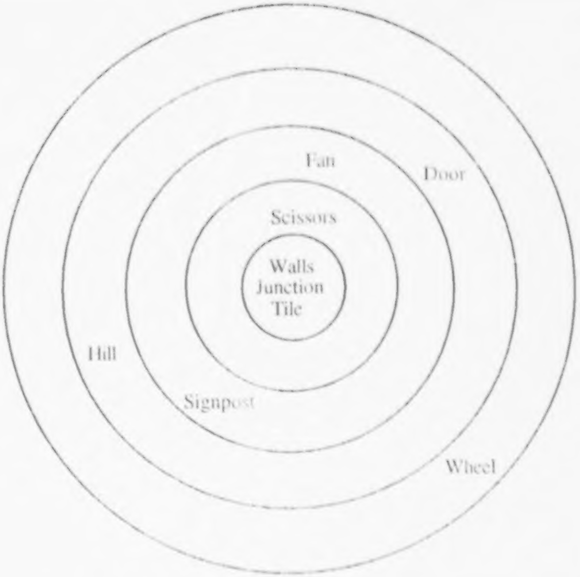

Figure 2: Modulating facet model of static similarity between nine situations

Table 4 shows the static similarity indices for all situation pairs, averaged over the three grade levels. The situations in Table 4 have been arranged in decreasing order of average static similarity to other situations—precisely the order in which the various situations are absorbed as one moves outward in Figure 2.

Table 4
Average similarity indices for each pair of physical angle situations

	Walls	Junction	Tile	Scissors	Fan	Signpost	Door	Hill	Wheel
Walls		0	0	0	3	0	0	0	0
Junction	94		0	0	3	0	0	0	6
Tile	95	94		0	0	0	0	0	0
Scissors	94	74	86		83	0	72	3	56
Fan	82	80	67	74		0	78	0	50
Signpost	77	76	76	63	59		0	0	0
Door	69	58	59	67	68	53		0	47
Hill	70	70	69	65	43	60	47		0
Wheel	57	53	41	62	62	51	53	38	

Note. Dynamic similarities are given above the diagonal and static similarities below.

Table 4 also includes average "dynamic similarity indices", calculated from Task 1 responses. It can be seen that the scissors, fan and door situations were regarded as dynamically more similar to each other than to the wheel. Only about a half of the students thought that the wheel turned like the other moveable situations. In fact, students often referred to the scissors, fan and door as "opening", not "turning".

DISCUSSION

The findings of this study call into question the received wisdom of defining an angle as an amount of turning. Firstly, many primary age students do not believe that all moveable angle situations involve the same turning movement. Secondly, students find it most unnatural to interpret a fixed situation in terms of any sort of movement. In neither case is there any change with age. One must conclude that "dynamic similarity" is not a sound basis for developing a general angle concept.

On the other hand, students do recognise a similarity between moveable angle situations and between moveable and fixed angle situations: they all involve two lines meeting at a point. This "static similarity" spans both moveable and fixed situations, and the recognition rate increases considerably between Grade 2 and Grade 6. Indeed, by Grade 6, students are as likely to recognise the static similarity between two moveable situations as the dynamic similarity.

A moment's consideration shows why static similarity should be more general, and therefore educationally more significant, than dynamic similarity. All physical angle situations *must* have two lines forming the arms of the angle, even if in some cases (e.g., signpost, door, hill, wheel) one or both lines must be imagined. On the other hand, there is no common attribute relating the two lines in different situations. For example, the angle between the two lines *may* represent an amount of turning but it can also show openness, sharpness, steepness, and so on. The angle concept depends on recognising that all these attributes are similar, in a special kind of way, and that is clearly more difficult than simply recognising common configurational elements.

The results of this study confirm our previous finding (White & Mitchelmore, 1997) that the main feature of a physical angle situation which affects angle recognition is the number of visible arms. The situation in the outermost partition of the facet model (Figure 2) has no visible arms; the situations in the next partition have one visible arm; and the situations in the three innermost partitions have two visible arms. In all the situations in the central partition, the two arms are readily identified; the arms of the scissors (in the next partition) are easily identified but move; and the arms of the fan and signpost (in the third partition) are embedded and not so easily identified.

In teaching about angle, one should probably aim to work from the central partition outwards. A first idea of angle could be easily established, even as early as Grade 2, from situations with two clearly visible arms—provided the examples are not restricted to right angled corners. The next step would be to seek similarities between these situations and fixed and moveable situations where both arms of the angle are present but not so easily identified (the next two partitions of our model). Students could compare the sizes of such angles without any reference to turning, possibly using a term such as "amount of opening" to denote the similarity. Each situation in the outermost two partitions, where one or both arms of the angle have to be imagined, presents its own problems and should probably be treated separately in later instruction. In each such situation, the main aim would be to construct the "missing" lines so that the similarity to other angle situations could be made explicit.

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CULTURE, GAMES AND MATHEMATICS EDUCATION: AN EXPLORATION BASED ON STRING GAMES

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ABSTRACT

This paper describes the research work in progress which is looking at the relation between cultural games and the teaching and learning of mathematics. The main aim of the research is to look at a number of games which are encountered in certain cultural settings with a view of exploring the use for these in mathematics classrooms. This paper focuses specifically on "String Figures", a game which is played by many people from different cultures in South Africa and other parts of the world. The paper describes the game and reports on the trial of the game in the classroom situation and a follow up of the trial. Results show a greater percentage of the students taking an active part in the game and subsequently in the lesson. However, it also shows that students do not easily see the relation between games and related mathematical concepts. The implications of these results are discussed and suggestions for alternative approaches in the use of the strings figures are made.

INTRODUCTION AND BACKGROUND

Games and mathematics teaching and learning

Games have been used in the teaching and learning of mathematics over the years, and this use has increased in the recent past (Vithal, 1992:178-179). Some of the uses of games in mathematics education are to learn the language and vocabulary of mathematics; develop mathematical skills, develop ability with mental mathematics; devise problem solving strategies; be the generator of mathematical activity at a variety of different levels; serve as a source of investigational work in mathematics (Kirkby, 1992; Sobel and Maletsky, 1975, Gardner, 1969). Many studies which explore the use of games in problem solving situations have been documented (Ecker, 1988; Gardner, 1978; Horak, 1990; Krulik, 1977; Krulik & Rudnick, 1989; May, 1993; Williford, 1992; etc), suggesting an important relation between games and problem solving, one of the central aspects in mathematics learning. The use of games also leads to discovery of patterns (Branca, 1974; Harlos, 1995; Oldfield, 1991), decision making (Buckhiester, 1994), and logical reasoning, deduction and skills (Brumfiel, 1974; Chandler, 1974; Ehrlich, 1974; Haggard & Schonberger, 1977). An important question that follows as a result of the different focuses of these studies is whether the uses of these games are applicable to the games in general or whether they are dependant on the type of the game. Specifically, the question that this study attempts to answer is whether games which are cultural or specific to a certain cultural group leads to the same kinds of outcomes, particularly as most the studies referred to above all deal with games that are not viewed as cultural specific.

The uses of games mentioned above suggest that games play a more profound role than just being recreational or pastime activities. As people engage in any game the language, vocabulary, mathematical skills and a lot of mathematical activities are generated. Although some of these may be very basic, they however serve as an important component of the development of mathematical concepts. Students may also discover mathematical concepts as they engage in games and puzzles (Sullivan, 1995).

CULTURE AND MATHEMATICS EDUCATION

The idea about culture as a key theme for mathematics education is coming up in various places (Mellin-Olsen, 1985). Most of the research on ethnomathematics focusses on these relations, particularly investigating how culture impacts on mathematical understanding. Masingila and Jamie King (1997) refer to two areas that researchers in ethnomathematics have tended to examine how people learn and use mathematics, the area of distinct cultures and the area of everyday situation within cultures. This study is located more in the second area of everyday situations within cultures, exploring the notion that various cultural activities contribute and enhance the understanding of mathematical concepts. The study does not look at all the different activities within a particular culture but rather focuses on the game of strings and the form in which it is played in certain cultures in South Africa, particularly among the Africans where the game is mostly played. Irrespective of the focus of the different studies, culture is seen to be playing an integral component in the understanding of and the development of mathematics.

Bishop (1988:182) argues that there are six fundamental activities which are universal in the sense that they appear to be carried out by every cultural group ever studied. He says that these six fundamental activities are necessary and sufficient for the development of mathematical knowledge. The six activities are (i) Counting, (ii) Locating; (iii) Measuring; (iv) Designing, (v) Playing and (vi) Explaining. The fifth activity of playing has to do with devising and engaging in games and pastimes, with more or less formalised rules that all players must abide by. In his study of the mathematics in the Mende culture, Bockarie (1993) found that the mathematics in this culture can be described broadly under six topics: (i) Counting and computation (ii) Ratios and fractions (iii) Forecasting and estimation (iv) Cultural values attached to certain numbers (v) mathematical games and (vi) mathematical applications. There are a number of common activities that both authors have identified as carried out by the different cultural groups. One of these is playing which is a central activity in games.

The definition of culture in this paper is drawn from the one used by Moschkovich (1995)which is "a set of practices, beliefs, customs, and institutions associated with a group of persons who are engaged in social activity (ies) together or associated with a social activity". The social activity in the context of this study would be string games which is examined in a mathematics classroom. Cultural games, drawing from the definition of culture by Moschkovich, would therefore mean those games that are played by a specific group of people who are engaged in a social activity together. The rules of a cultural game are most likely to be understood better by people of that culture, and may be understood and interpreted differently by people of a different culture. For example, Gerdes (1994:351) refers to the 'struggle for territory' game which is very popular with the Egyptians and may be understood differently by people who are not of Egyptian origin. The same applies to the 'Mu Torere game' which is played by the Maori people in New Zealand (Zaslavsky, 1996:187). The argument is not that these games cannot be understood by other people. On the contrary, they may be even better understood by people of a different culture who have given themselves time to thoroughly learn the rules of the game. However, this game would not have the same kind of meaning and understanding, possibly not even the same interpretation for the two groups of people.

STRING GAMES

String games are popular with children in many parts of South Africa, many African countries (see Lindblom, 1930) , and many other countries in other parts of the world. Although string figure games are popular in different parts of the world, the emphasis seems to differ from country to country, or more specifically from culture to culture. Whereas in a particular culture emphasis may be on the making of gates, in another the concentration is on different figures which may be made using strings. Knowledge of the game among the different children usually comes from the interaction among themselves, engaging in different cultural activities, one of which is games. The game described in this paper is usually played by children of school going age, especially at the primary level of schooling. By the time the children start going to school, some of them have already come across the game as they interact and play with their peers. The game involves using a piece of string - length of 1metre adequate, a smaller length would make the gates very small which are not easily recognizable, especially as the number of gates increase to more than three- to produce different number of 'gates' (this is the name used for the game in some parts of the country, particularly in the townships. The name varies from place to place. In some places it is known as 'Diheke' - this is the name used mainly in townships, whereas in other places, particularly people who speak the Setswana language it is known as 'Malepa'). The 'gates' or 'diamonds' are then used to find patterns that result from the geometrical shapes (quadrilaterals and triangles) made by the string. This seems to be the most popular application of string figures. Amir-Moez (1965) gives examples of how string figures can lead to generalisations in mathematics as well as mathematical induction.

Methodology in the trial on string games

Three secondary schools in the vicinity of the University of The North in the Northern Province of South Africa were used for the purpose of the initial trial. These three schools were selected on the basis of involvement in an ethnomathematics project which had been running under the auspices of the RADMASTE Centre at the University of The Witwatersrand (Amoah, 1996; Laridon, 1995; Mosimege, 1995). Either the teacher or the researcher (some of the teachers asked the researcher to conduct the lesson) started the lesson by introducing the game to be played in class, then distributed the material to the students which they used to play the game.

At two of the secondary schools the teachers requested the researcher to continue with the lesson as they were not any longer very familiar with the use of the worksheet on string figure patterns. I gained an impression that the teachers had never looked at the material since the last workshop (this workshop had taken place sometime in 1995, and this trial in the schools took place one year after the workshop), making the activities to be relatively new to them and this was likely to create an embarrassment in the presence of the researcher. At the third school the teacher conducted the lesson but asked the researcher for assistance whenever there was a need. The procedure followed in class was as follows:

-each student was supplied with a piece of string and the worksheet on string figure patterns;

-all the students were given a few minutes to try any of the string figure patterns that they were able to do;

-the researcher then called on any of the students to demonstrate any of the string figure patterns that they could do to the whole class, doing those that none of the students could do;

-when this demonstration was done the rules of making a specific number of 'gates' were explained to the whole class by the researcher and then the students were given some minutes to try the gates again, following the demonstration (this was intended to help those students who didn't know the game how to play the game).

All the students at the Secondary Schools who participated in this activity were in Standards 6 and 7 (Grades 8 and 9). The classrooms were selected on the basis of the knowledge that String Figures lead to identification of geometrical shapes which are encountered in these standards. At the end of the activities the students were given a questionnaire on the use of games in mathematics to complete.

RESULTS OF THE INITIAL TRIAL AND DISCUSSION

All the children showed interest in playing with the strings, although some of them had not played string figure games before - a greater percentage of the students were able to make some gates, although most could do only two or three gates. The mathematics concepts that the students mentioned were angles; triangles; and parallel line which are formed by the top and the bottom part of the string after the gates have been made. However, it was clear that many of the students did not clearly see the relation between mathematical concepts and the games. Even in those instances where students were quite familiar with the game, it was still very difficult to identify related mathematical concepts. In certain instances, the researcher had to question the students on geometrical shapes that they had studied in their mathematics classrooms, and lead the students through a series of questions so that they could ultimately see the relations. Further analysis of the students' responses suggest that they see games as something separate from the mathematics they do in the classroom, leading to the difficulty in identifying geometrical shapes that they had studied before.

TABLE 1
SCHOOL A (N=153)

QUESTION	STANDARD 6 (N=86)		STANDARD 7 (N=67)	
	YES	NO	YES	NO
1. Ever played a game in the mathematics classroom?	8.14% (7)	91.86% (79)	31.35% (21)	67.65% (46)
2. Did you enjoy playing the game today?	100% (86)	0%	100% (67)	0%
3. Did you understand the game and its rules?	30.23% (26)	69.77% (60)	47.76% (32)	50.75% (34)

TABLE 2
SCHOOL B (N=29)

QUESTION	STANDARD 6 (N=20)		STANDARD 7 (N=9)	
	YES	NO	YES	NO
1. Ever played a game in the mathematics classroom?	55% (11)	45% (9)	22.22% (2)	66.66% (6)
2. Did you enjoy playing the game today?	100% (20)	0%	100% (9)	0%
3. Did you understand the game and its rules?	25% (5)	70% (14)	44.4% (4)	44.4% (4)

TABLE 3
SCHOOL C (N=102)

QUESTION	STANDARD 6 (N=54)		STANDARD 7 (N=48)	
	YES	NO	YES	NO
1. Ever played a game in the mathematics classroom?	25,93% (14)	74,07% (40)	35,42% (17)	64,58% (31)
2. Did you enjoy playing the game today?	100% (54)	0%	100% (48)	0%
3. Did you understand the rules of the game?	29,7% (16)	70,3% (38)	47,92% (23)	52,08% (25)

A higher percentage of most of the children in all the standards except in standard 6 in school B (55% indicated that they had played games before) responded that they had not played a game at all in the mathematics classrooms. This situation is a cause for concern particularly in the context of the use of games as recorded in the literature. It is important to note that a description of a game had not been given to the students prior the questions on games, and as a result the students may have not necessarily had the same understanding of the meaning of a game. All the students in the study indicated that they had enjoyed the string figure games that we had played in the classroom together.

The majority of the students in all the schools indicated that they had not understood the rules of the games. When I asked a few students about understanding the rules of the game I discovered that most of them equated understanding the rules with having played the game before or being able to successfully make many gates with the string. This highlighted one of the difficulties that may accompany using games in the classroom, in which students may confuse

Most of the games that the students listed when asked to mention any game that they knew that could also find use in the classroom were games like Monopoly, Maths 24, Snake and ladders, Get Four,etc. These games are not necessarily cultural in nature in the same way as we could refer to a game which is mainly known and played in a particular cultural setting. They are games that students have played before, especially with friends outside the mathematics classroom setting. A few cultural games like Moruba (a cultural game very popular in the Northern Province), and Morabaraba were also mentioned.

It had been planned that at the end of the game the students would be given a worksheet on string games to complete. However, the amount of time spent on the game itself was such that there was not time left for this activity. This is another of the important outcomes which highlighted the fact that in order for games to serve the desired purpose, there needs to be enough time beyond the usual one or two 30 minutes periods allocated for mathematics lessons in a day.

RESULTS OF A SECOND TRIAL AT SCHOOL A AND DISCUSSION

One year after the initial trial i.e. 1997 the first school was visited to conduct a follow up exploration in the classrooms. The same teacher who was involved in the first trial was asked to conduct the lesson (in the first trial the lesson was conducted by the researcher). The trial was conducted in two Standard 6 classes which were taught by the teacher - these were the classes which the teacher had been allocated to teach for the year and another teacher was allocated to teach the classes that this teacher taught previously. The classes were taught on the same day, giving the researcher an opportunity to observe any difference in the way the teacher approached the lesson and improved on the ability and expertise of showing the different gates. This situation however deprived the researcher of the opportunity of comparing the same students to determine if they could improve on their performance in the same game.

TABLE 4
SCHOOL A

QUESTION	STANDARD 6 A (N=62)		STANDARD 6 B (N=67)	
	YES	NO	YES	NO
1. Ever played a game in the mathematics classroom?	59,68%(37)	40,32%(25)	52,23%(35)	47,77%(32)
2. Did you enjoy the game today?	100% (62)	0%	100% (67)	0%
3. Did you understand the rules of the game?	32,26%(20)	67,74%(42)	41,79%(28)	58,21%(39)

A comparison with the previous students at the school indicate an increase in the number which had played games in the mathematics classrooms. The teacher ascribed this to the introduction of some games in her mathematics lessons, which she did as a result of the active involvement which she had noted in the previous year's research. However, the students still seem to be equating the understanding the rules of the string game with the ability to make many more gates. This emerged when most of the students were able to make two or three gates but found making more gates difficult.

IMPLICATIONS FOR THE CLASSROOM

Games can play a significant role in the teaching and learning of mathematics. The fact that ultimately, after some probing, the students were able to mention some mathematical concepts that were related to the game, showed that there is a definite use for games in mathematics lessons. However, the use of games should be implemented with care. This means that the choice of the game should be done with a specific purpose in mind. It is necessary that the game first be analysed for its mathematical content so that these can be targeted when the objectives of the game are set. However, if the analysis does not necessarily reveal any immediate mathematical content, this does not make their use in the classroom any less important. As May (1993) expresses it, the students may use the games to plan ahead for different mathematical tasks.

When games (irrespective whether they are cultural or not) are played in class, attempts should be made by the teachers to go beyond the level of mere enjoyment of the game. The students tend to be more engrossed in the playing than the learning process, except for a few who see the intended lesson. When some of them are questioned at the end they can hardly relate or recall what they have learned. However for some, the relations between games and mathematical concepts are very apparent, and this seems to be the case with those who had a thorough understanding of the game before the lesson.

students tend to be initially focussed on not losing the game and in the process may miss out on the advantages of the use of a game. She continues to say that all students at times are inefficient in monitoring conclusions and generalisations they made while playing the game. This is likely to result from the use of games if the teachers do not review and change their roles accordingly whenever games are used in mathematics classrooms.

One of the difficulties that emerged in this study is that it was not possible to get to the worksheet on string games and subsequently an exploration of other mathematical concepts like symmetry and induction which could be explored in the string game. Apart from the implications in terms of availability of more time for games in general, it also suggests that another approach could be tried to derive more benefit out of these games. The fact that some of the students were able to demonstrate to their fellow classmates some of the gates implies that they have an understanding of the game. A possible exploration in another related study could be to investigate how students explain what they know to others. This would remove the constraint of imposing personal understanding on these games and bring in the students' understanding and mathematical development.

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Enlarging Mathematical Activity From Modeling Phenomena to Generating Phenomena

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Traditionally, mathematics has been used as means for modeling aspects of the experienced world, and it is often taken as axiomatic that one can learn mathematics more effectively if one is able to apply what one already knows and can do. We illustrate how we can substantially deepen the connection with everyday experience by using mathematical functions to generate phenomena as well as model them. We first provide a framework for examining relations among simulations, notations and physical phenomena. We then illustrate with a 9th grade classroom episode how students' activity taps into their linguistic, kinesthetic and notational resources to deepen their engagement with important mathematical ideas.

Introduction

This paper builds upon prior work (Kaput & Roschelle, 1997; Nemirovsky, Tierney, Wright, in press) to which the reader is referred for background and particular features of the learning environments that we are using. For the reader's convenience we quote an edited abstract from Kaput & Roschelle (1997):

> We address the question of how we might exploit interactive technologies to democratize access to ideas that have historically required extensive algebraic prerequisites. Illustrations will be drawn from work in the authors' ongoing SimCalc Project, which builds and tests software simulations, physical devices, and related visualization tools intended to render more learnable the ideas underlying calculus and the Mathematics of Change & Variation beginning in the early grades. We will reflect on how such technologies can change the experienced nature of the subject matter by tapping more deeply into students' cognitive, linguistic and kinesthetic resources. Substantial reorganizations are possible of curricula that have been taken as given for centuries (p.105).

Our purpose here is to focus on an important new affordance of technological learning environments, the ability to *generate* and not merely to *model* phenomena. These environments enable the user to embody their intentions and conjectures within phenomena that he/she can control and interact with in new ways. For example, a student can use mathematical notations or other controls within a computer environment to control the motion of physical "Minicars" that move on tracks. The same means can also drive a simulation "within" the computer. Our goal is to understand how to use the rich learning opportunities that emerge as we combine these 2 kinds of phenomena-generating capabilities with each other and with traditional affordances such as linked representations & MBL devices.

We will first briefly describe the theoretical and technological contexts for our work, and then sketch a classroom episode that illustrates the fundamental issues that emerge in such learning environments. An enlarged version of this paper will be available at the PME-22 Meeting, as will videos of the episode and demonstrations of the devices described.

Activities Crossing Notational, Simulation & Physical Realms

Historically, we have always assumed that mathematics was to be used to model and make sense of situations and phenomena - in Yerushalmy's (1997) words "thinking about one thing in terms of simpler, artificial things" (p.165). The thing to be modeled is taken as having an existence independent of the model. But, as we turn to computer-based learning environments, we need further distinctions.

1. Phenomena "inside the computer" vs physical phenomena "outside the computer."
2. "Target phenomena" taken as the subject of mathematical description or control (e.g. motion) vs "notational phenomena" embodied within notations that may be used to describe or control the former.

Clearly, distinctions between targets and notations are relative to intentions at hand. We are using them for heuristic purposes to help expose structures of activities that involve coordination across the different realms of phenomena reflected in Fig. 1, whose arrows refer to electronically realizable connections. This diagram points to an extremely rich set of activity-possibilities while simultaneously omitting some of the most important features of all of them, such as: 1) human interaction: talk, gesture, reflection, conjecture, imaging, comparing, explaining, etc. - the medium in which the activities pointed to come alive. And 2) a related set of "invisibles" - the resources that students bring to the activities, the at-hand results of their everyday experience with language, symbols, space, time, objects, motion, and so on.

We now examine the structures of activities that the diagram points to, and in so doing contextualize some traditional work involving computer technology and reveal some special features of our own work.

Expanding Realms of Activity-Structures

Expanding linked representations in the notational realm to Rate-Totals links

Many researchers, too numerous to mention, have given attention over the past 10 years to the use of multiple linked representations of functions. In terms of Fig. 1, this work concerns the inside of the upper left circle, although in practice much of the work also involved modeling of situations, usually given independently from the computer environments in text form, an exception being Yerushalmy (1997), some of whose software incorporated text fragments as a fourth linked notation system.

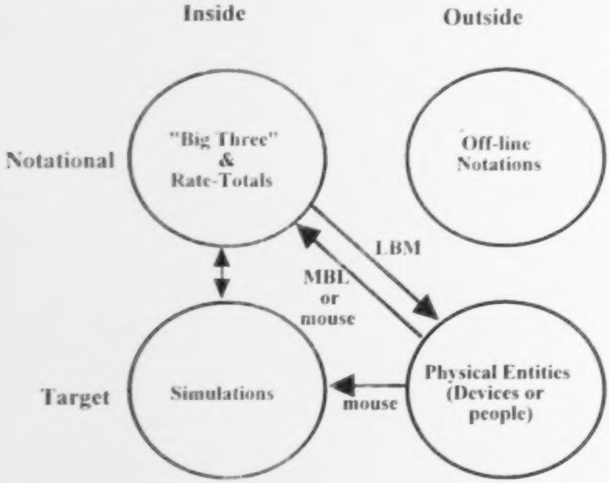

Figure 1

Within the linked representation view, we take the perspective that *understanding the relations between rates and totals descriptions of varying quantities* (and the situations or phenomena that they describe), is a fundamental aspect of quantitative reasoning as it relates to the MCV. For example, to understand the difference between linear and quadratic growth as traditionally approached, one looks at the shapes of the respective graphs, or the growth patterns in their respective tables, in relation to the corresponding equation. But from our perspective, the linear and quadratic functions must also be seen as related by the rates-totals connection. The quadratic is the accumulation of linearly changing quantities, and the linear is the rate of change of quadratically changing quantities. We believe that these kinds of connections allow a much deeper understanding of the basic mathematics than do approaches that ignore them. We not only can connect graphs and formulas, we can cross-connect, for example, a rate graph to a totals formula. Physical realizations of rate-totals connections can take many forms, but we take the historical starting point, the velocity-position connection in motion phenomena.

Expanding the realm of simulations to MBL Data

Simulations involve the top-down arrow in Fig. 1. Such a graphically-oriented software context, called MathWorlds (http://www.simcalc.umassd.edu), was discussed in Kaput & Roschelle (1997). A somewhat more formula-oriented environment, is called AlgebraAnimator (Logal, 1995). And over the past two decades Microcomputer Based Laboratory (MBL) devices (Tinker & Thornton, 1994), have become more common, and are represented by the arrow from the lower right to the upper left in Fig. 1. We and our colleagues have made extensions of this functionality to enrich the work with simulations (see Fig. 5 in Kaput & Roschelle, 1997) to enable the student both to increase the intensity and personal intimacy with the mathematical representations (Middleton & Kaput, in preparation) as well as to create new situations such as parades or dances that embed function-descriptions of motion in interestingly complex relations. In Fig. 1 these constitute an extension bridging the lower right circle with the two on its left.

Expanding the realm of physical action and connections among notational, cybernetic and physical phenomena

Nemirovsky (1993) has explored numerous ways to expand the realm of physical action by designing devices that allow students to control air flow, the shape of a surface, or rotary motion - either directly by acting on the physical device, *or by using a mathematical function defined on a computer that "drives" the device.* This functionality, now common in industrial manufacturing situations, is represented by the upper to lower-right diagonal arrow labeled "LBM." ("Lines Become Motion.") These devices have been used for investigating how students, explore, think and learn about ideas of rate and accumulation (Nemirovsky and Noble, 1997).

This phenomena-generating capability turns a fundamental relationship between mathematics and experience from one-way to bi-directional, which in turn supports a much tighter and more rapid interaction on which to base learning. Because the mathematical notation that controls a phenomenon also models it, one can test a model immediately, as we see below. Critically, the student's intentions can be made visible, explicit and testable through the phenomena that the student controls.

Our major goal is to understand how the two forms of phenomena-generation activity - simulations and physical devices - can best be used in combination and in different kinds of classroom situations, optimally exploiting the strengths of one to compensate for weaknesses in the other, as well as build on mutual strengths.

Case Study Illustrating Interactions Among Physical, Cybernetic and Notational Spheres of Activity

The episode took place in the 9th grade algebra class of Michigan, USA teacher, Kellie Bachman, who was participating in a series of experimental activities directed by her colleague Marty Schnepp. In front of the classroom was a car on an inclined plane (Fig. 2 A - upper) connected via MBL to a computer that displays its velocity in real time (Fig. 2 B), and an LBM "Minicar" (Fig. 2 A - lower) whose motion can be controlled by a graph drawn on the same computer screen.

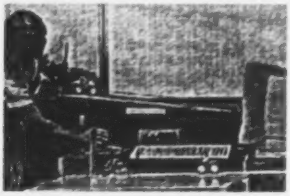

Figure 2 A

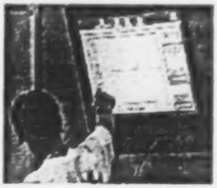

Figure 2 B

The Episode

We will briefly describe the first 30 minutes of the session. The teacher began the class by reviewing a homework task, which consisted in creating qualitative graphs describing the speed of a sled going up and down a hill. She drew 3 different graphs that students had proposed:

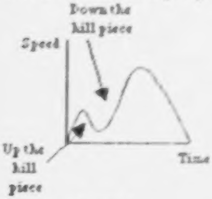

Figure 3

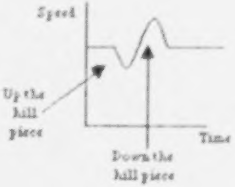

Figure 3

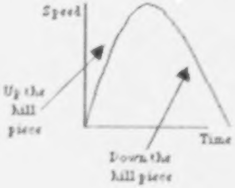

Figure 4

The teacher proposed to measure the speed of the car on the inclined plane to generate a graph that would correspond to the "downhill" part. The graph that appeared on the computer screen as the car rolled down was similar to Fig. 6 (the car bounced back a few times as it reached the end of the inclined plane). The immediate discussion focused on the negative velocity peak. Students commented: "it hit a tree or something," "it bounced back." Students proposed some experiments to verify that the meaning of negative velocity was "going backwards." The first one was to move the car by hand to see whether the velocity becomes negative. Then they wanted to see if they could "catch" the car by hand on its way down, preventing it from bouncing upwards. This proved to be difficult because one tends to stop the car rolling down by slightly pushing it upwards; this subtle effect that became apparent in the resulting velocity graphs with short negative sections.

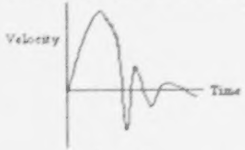

Figure 6

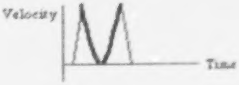

Figure 7

Right after one student succeeded in stopping the car without producing negative velocity, the class discussed small "bumps" that appeared on his velocity graph after the car had been "stopped." Another student explained the bumps: "it is because he stopped it, let it move [the car], stopped it, let it move, until it [the car] got to the bottom [of the inclined plane]." Students accompanied these descriptions by moving their hands in the manner they thought had generated the small bumps. To see more clearly the effect of these small bumps, the teacher proposed to move the Minicar driven by the same velocity graph that included the "little bumps," transforming the velocity *model* to a velocity *generator*. Because the motor makes a sound whose pitch is proportional to its speed, the teacher advised to "listen carefully to the sound of the motor." (Variations of speed that are difficult to see are often easy to hear). The Minicar reproduced in its motion the "bumpy" slow motion that the student had produced by hand to let the car reach the bottom of the inclined plane. Inspired by this experiment, students now wanted to make the Minicar reproduce the original motion of the car rolling down on the inclined plane when it "bounced back."

This was an opportunity for the teacher to discuss again the interpretation of Fig. 6. Since the students kept referring to this graph as the one where the sled had "hit a tree," she asked exactly at which point on the graph the sled had hit the tree. Some thought that it was at the low extreme of the negative peak, others when the graph crossed the horizontal axis. After running the Minicar and reaching an apparent consensus on the latter, the teacher asked whether they wanted to make other experiments. One student wondered how the Minicar would "do" if one drew a velocity graph for it that did not start at zero velocity. How would the Minicar realize such a request? They tried it and saw that even before running the car, the software added an initial piece to the drawn graph so that it would start from zero. The student who had proposed Fig. 4 for the homework said that a velocity graph could, however, start from a non-zero value; one could, say, start the graph when one is walking toward the hill with the sled. On the other hand "if you start from sitting at the bottom of the hill I agree" that the graph had to start at zero velocity.

The teacher then proposed to discuss Fig. 5. Several students said that the graph should be inverted (uphill slowing down and downhill speeding up). When the inverted graph was drawn on the computer to drive the Minicar, the software modified the "U" shape producing Fig. 7. At this point the class had no difficulty in interpreting why the software had "added" the initial and the final segment. After observing the Minicar enacting Fig. 7, two students made the following comments:

Jesse: When the graph is going up, the sled is going down. When the graph is going down, the sled is going up or something. That's confusing, because...

Ben: Just think about it as the speed going up, not the sled going up. We are talking about the speed and the time, not the sled. It has to do with the hill because it has to do with what part of the hill you are on, but it is really about the speed.

Analysis

Three aspects of this episode seem of particular relevance for our research question: *How can technologies and learning environments change the experienced nature of the MCV by tapping more deeply into students' cognitive, linguistic and kinesthetic resources?* These 3 aspects suggest some of the ways in which the interplay among notations, simulations, and physical phenomena can be productively expanded: by incorporating the kinesthetic activity, empowering notations to create phenomena, and testing similarities and differences between virtual and physical phenomena.

1. *Incorporating the kinesthetic activity.* The car rolling downward displayed its corresponding graph showing, for the first time in this class, the possibility of negative velocity. To account for this new possibility the students did a number of experiments moving the car by hand. The task of stopping the car without generating a negative velocity became an engaging *kinesthetic* challenge. Note how negative velocity developed a significance that went beyond the mere statement of "it means going backwards." It brought to the fore the common but ignored fact that the act of "stopping" a motion ordinarily generates a damped oscillation.

2. *Empowering notations to create phenomena.* Controlling the Minicar with the "bumpy" graph generated by hand not only confirmed that the velocity bumps represented slow and discrete motions, but gave to the graph, as a symbolic notation, a different status: it can create phenomena. The graph not only represented how the student had moved his hand, but it was also "empowered" to drive a physical car. An indication of this new relevance is the spontaneous student requests to run the Minicar according to other graphs that had been discussed before.

3. *Experimenting with similarities and differences between virtual and physical phenomena.* Even though this particular class was not using simulation software, there were pervasive "imaginary" simulations in their talk about the sled going up and down the hill, hitting a tree," sitting at the bottom of the hill," etc. The discussion of starting at non-zero velocity made prominent some of the differences between cybernetic and physical phenomena. While the students could imagine a velocity graph starting at non-zero velocity by beginning the graph as one "is already walking," the graph driving the Minicar – subject to the constraint that it must describe the *entire* motion of the car - is *forced* to start at zero velocity.

Overall, the students and teacher seemed fluently and easily to combine the three realms in Fig. 1 (the 4th, Offline Notations, appeared in the homework). They did not seem confused or troubled by referring to the car on the inclined plane as hitting a tree that was not there or by using the same graph to symbolize different actions. This seems to indicate that these tools and learning environments help to recruit linguistic and cognitive resources from their everyday experience.

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Young Students' Constructions of Fractions

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Grade 4 and 6 South African students' concepts of and operations with fractions were investigated using written tests. Their responses to some of the items were analysed in terms of success and misconceptions. In line with international studies, these students had several limiting constructions regarding fractions. Some of these may be interpreted as the result of the current teaching approach for fractions, while others might be the result of students' incorrect intuitions which have not been resolved or clarified in the classroom.

Introduction

This paper reports on the first phase of a project that addresses the learning and teaching of fractions in the elementary grades. The project involves the development and selection of materials and the in-service training of teachers. In order to inform this development, and to make evaluation of the success of the implementation of these materials and training possible, the project began with a base-line study of students' present understandings in four large government schools in the Western Cape, South Africa.

The results of this baseline study are interpreted within our existing theoretical framework of the learning and teaching of mathematics in general, as reported in previous PME papers e.g. Murray, Olivier & Human (1996). Such an approach is based on the view that students construct their own mathematical knowledge irrespective of how they are taught. Cobb, Yackel and Wood (1992) state: "…we contend that students must necessarily construct their mathematical ways of knowing in any instructional setting whatsoever, including that of direct instruction," and "The central issue is not whether students are constructing, but the *quality and nature* of these constructions" (p. 28, our italics).

There are many factors that may contribute towards elementary school students' poor understanding of common fractions. Based on the research results reported by, for example, Baroody & Hume (1991); Streefland (1991) and D'Ambrosio & Mewborn (1994), as well as local projects e.g. Murray *et al.* (1996), there appear to be three main possible causes:

- The *way and sequence* in which the content is initially presented to the students, in particular exposure to a limited variety of fractions (only halves and quarters), and the use of pre-partitioned manipulatives;
- A *classroom environment* in which, through lack of opportunity, *incorrect intuitions and informal (everyday) conceptions of fractions are not monitored or resolved*; and

- The inappropriate application of *whole-number schemes*, based on the interpretation of the digits of a fraction at face value or seeing the numerator and denominator as separate whole numbers. This can be seen as a special case of the previous problem, but is commonly reported in the literature and will thus be considered separately.

This paper reports on our investigation of our first hypothesis, namely that if these were the main causes, they would be evident in our analysis of students' present understandings of fractions.

Methodology

In South African primary schools, fractions are usually introduced by presenting halves and quarters using pre-partitioned geometric shapes or other manipulatives. It is expected that by Grade 4, students should have been introduced to selected fractions in the mathematics classroom in this manner. By the beginning of Grade 6, according to the syllabus, students should have been exposed to equivalent fractions, comparisons of the sizes of fractions, and then addition and subtraction of fractions.

Thus Grade 4 and Grade 6 were selected as important age-groups at which students' current understandings of fractions should be investigated in the base-line study. Written tests were designed to evaluate these students' concept of what a fraction is (Items 1 to 3 in Table 1), comparison of the size of fractions with different denominators (Items 4 to 6) and their operations with fractions (Items 7 to 13). It can be seen from Table 1 that the tests for Grade 4 and Grade 6 had several items in common, in order to investigate the effect of age and teaching. Although multiplication and division with fractions are not covered in the syllabus at either level, such items were also included in the Grade 6 test, in particular division items that challenge common experience-based ideas, e.g. that 'division makes smaller'.

The tests included context-free items and items in context. Some items were adapted from previous studies (e.g. Baroody & Hume, 1991; Pirie & Kieren, 1992; D'Ambrosio & Mewborn, 1994). The items were refined and adapted after an analysis of students' responses in a pilot study in three schools.

370 Grade 4 students and 382 Grade 6 students participated in this study, representing all three main language groups in the Western Cape, namely Xhosa, Afrikaans and English. Tests were available in all three languages. The tests were administered and coded by the Fractions Working Group of the Mathematics Learning and Teaching Initiative.

Results

The following table gives a summary of the success rate on each of the items, as well as the most common misconceptions identified.

Item	Success[1] Grade 4 (n=370)	Success Grade 6 (n=382)	Most common misconceptions (Percentages given as Grade 4; 6)
1. What does $\frac{4}{5}$ mean[2]?		15%	a) 9, 20 or 'four fives' (3%); b) 'recipes': numerator, denominator (7%) c) shaded geometric shapes (7%)
2. Show $\frac{3}{4}$ in at least 3 different ways.	11%	14%	a) 12; 1; 7; 'three fours'; 3-4 (or 4-3); 3+4; or 3×4 (8%; 3%) b) incorrect shaded geometric shapes (12%; 10%)
3. Mother has 10 smarties[3]. She says you can have $\frac{3}{5}$ of the smarties. How many smarties will you get?	2%	10%	
4. Put these fractions in order from smallest to biggest: $\frac{2}{5}$, $\frac{2}{3}$, $\frac{2}{9}$	5%	23%	a) order $\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{9}$ (16%; 38%) b) 2;2;2;3;5;9 (14%; 3%); c) $\frac{6}{17}$ or other whole-number procedures (22%; 5%).
5. Would you rather have $\frac{3}{5}$ or $\frac{3}{4}$ of a pizza? Why?	10%	26%	a) $\frac{3}{5}$ is larger (18%; 30%)
6. Jean spends $\frac{1}{4}$ of her pocket money. Piet spends $\frac{1}{2}$ of his. Could Jean have spent more than Piet? How?	2%	3%	a) No, $\frac{1}{2}$ is bigger than $\frac{1}{4}$ (22%; 33%) b) $\frac{1}{4}$ is bigger than $\frac{1}{2}$ (8%; 18%)

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[1] In the cases where the cell is blank, the item concerned was not given to students in this grade.

[2] For Items 1 and 2, the percentage given is the percentage of students with at least one correct response.

[3] Smarties are small, round candy-coated chocolates.

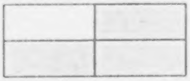
Item	Success Grade 4	Success Grade 6	Most common misconceptions
7. $\frac{7}{8} + \frac{7}{8}$	12%	31%	a) 30, '15+15', '14+16' or '14,16' (25%; 9%) b) $\frac{14}{16}$ (2%; 32%) c) $\begin{array}{l} {}^{1}\,\frac{7}{8} + \frac{7}{8} \\ \hline \underline{1\ 6\ 5} \end{array}$ (2%; 2%)
8. $\frac{2}{3} + \frac{4}{5}$		11%	a) $\frac{6}{8}$ (43%) b) Partial procedure for finding equivalent fractions, e.g. $\frac{6}{15}$ (5%) c) 5 as the LCD, e.g. $\frac{6}{5}$ (5%)
9. $\frac{3}{4} \times \frac{2}{5}$		38%	a) Equivalent fractions procedures not leading to successful solution (12%)
10. $2 \div \frac{1}{2}$		8%	a) 1 or $\frac{2}{2}$ (21%)
11. $4 \div 8$	5%	7%	a) 2 (32%; 50%)
12. Some friends go to a restaurant and order 3 pizzas. The waiter brings them the pizzas, sliced into eighths. Each person gets $\frac{3}{8}$ of a pizza. How many people will get pizza?	12%	11%	
13. We need $\frac{1}{2}$ metre of material to make a scarf. How many scarves can we make if we have 2 metres of material?	20%		

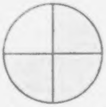
Table 1: Success Rates and Misconceptions by Item: Grade 4 and Grade 6 Students

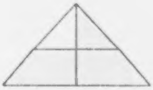
Discussion

It was not always possible to identify common misconceptions, as in many cases students omitted items or responded with various fraction or whole-number answers with no explanation.

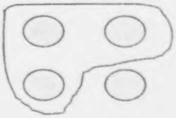
Misconceptions arising from initial exposure to fractions at school Some students produced memorised 'recipes' for fractions, for example "numerator, line segment, denominator" (1b). There was also evidence of the reproduction of pre-partitioned illustrations (1c, 2b). For example, some students responded to Item 2 by drawing the following three shapes:







Some students even wrote "square, rectangle, circle", particularly in one class where such an illustration of fractions was displayed on the wall. The generalisation of the partitioning of a shape into four parts to a triangle indicates a limited concept of a fraction that does not include equal partitioning. The introduction of fractions using mainly a continuous area model in which the fraction represents part of the whole[4], was also evident from the fact that while 20% of Grade 4 students and 35% of the Grade 6 students produced such illustrations of shaded geometric shapes, only 2% of Grade 4 students and 3% of Grade 6 students represented $\frac{3}{4}$ as a part of a collection of objects, for example:



The 'smartie' problem (Item 3) which addresses this particular meaning of the fractions, had a particularly poor success rate, especially at Grade 4 level (2%). It is probable that the students had not previously been exposed to problems that address the fraction as part of a collection of discrete objects.

The poor success rate on Item 6 (2%; 3%), in which students did not consider the role of the whole (6a), could also be interpreted as the consequence of being exposed to a limited range of problems. In this case, the problems that the students have solved in the past probably required them to compare the size of different fractions in cases where the whole is *always the same* and is usually a single continuous shape. Similarly, exposure to division problems which can *always* be interpreted as *sharing* and in which the divisor is thus always a whole number and always smaller than the dividend, may also be a cause of the poor success rate on Items 10 (8%) and 11 (5%; 7%).

The results also suggest that some of the students had been exposed to a procedure for generating equivalent fractions, but with little understanding of the purpose of this procedure (9a) or the reasoning behind it. The latter resulted in some students remembering only part of the procedure, namely finding the lowest common denominator (8b). Other students took 5 as the lowest common denominator (8c). The latter can be interpreted as the result of the traditional sequence for teaching addition of fractions: first addition with like denominators, then with denominators of which one is a multiple of the other (of which one then chooses the larger), and finally with unlike denominators. In this case, students may have generalised their current rule (for the second type of problem) to [illegible] unfamiliar situation of fractions with unlike denominators.

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[4] It is of course true that the given responses might not be the students' *only* interpretation of fractions, but the first or easiest response or that which the students believe is expected of them.

Misconceptions arising from students' own incorrect intuitions and informal experiences Students' inability to interpret the item "$2 \div \frac{1}{2}$" as 'how many $\frac{1}{2}$'s are there in 2?' provides an example of a limiting construction arising from their own intuitions and real-life experiences. The division of fractions out of context conflicts with students' deep-seated ideas about division, as it produces an answer larger than the number to be divided and, unlike whole-number division, cannot be interpreted as a 'sharing' situation (Baroody & Hume, 1991). However, exposure to a wider variety of division situations at school would have provided an opportunity for this conflict to be resolved, as mentioned above.

Inappropriate application of whole-number schemes One of the most common general errors identified in this study was the students' inability to see a fraction as a quantity - a quotient relation between two numbers – rather than two separate whole numbers. In fact this is an example of a misconception or a limiting construction based on students' own intuitions and previous experience. In this study, it affected the students' success on the items investigating their concept of a fraction, comparison of the size of fractions and addition of fractions. Students responded to the face-value of the denominator and numerator, separating them and carrying out inappropriate operations with them (Misconceptions 1a, 2a, 4b, 4c). When it came to adding fractions, some students mechanically combined the two denominators and numerators (7a, 7b, 7c, 8a). It is interesting that the application of a similar whole-number scheme leads to success when it comes to multiplying fractions (Item 9). The interference of whole-number strategies, which results in students simply adding the numerators and the denominators, has been widely reported in the literature (Baroody & Hume, 1991; Streefland, 1991; D'Ambrosio & Mewborn, 1994). Even if students had not previously been exposed to the addition of fractions, it is disturbing that they were unable to judge that 30 and 165 are not reasonable answers for "$\frac{7}{8} + \frac{7}{8}$" (Item 7).

Comparing the size of fractions by considering only the size of the denominator can also be considered a case of regarding the numerator and denominator as two unrelated whole numbers (4a, 5a, 6b). This misconception was actually more prevalent in the case of the Grade 6 students than in the case of the Grade 4 students, although the success rate on these problems increased across these grades. The tendency to choose as the larger fraction the one with the larger denominator has also been reported in the literature (e.g. Baroody & Hume, 1991).

Also in the case of division by a fraction, students responded by attempting to apply a previous whole-number scheme (10a). Similarly, changing $4 \div 8$ to $8 \div 4$ may be a misapplication of the commutative law that applies in the case of addition but not in the case of division (11a).

In summary This research has found that, in line with research in other countries, these students have, after their first few years of school, limited and *limiting* understandings of fractions which persist into the upper elementary grades. These meanings could in some cases result from students' own intuitions or might be a direct result of aspects of the teaching approach currently used for the introduction and development of the fractions concept in these schools. In both cases the findings presented here suggest that the current teaching approach has not been successful in challenging students' incorrect intuitions and in preventing the development of additional limiting constructions.

Testing our approach to teaching and learning fractions Although this study suggests that students have some intuitions and experienced-based ideas which are incorrect, there is evidence that very young children are able to understand and solve sharing problems involving fractions (e.g. Empson, 1995; Murray *et al.*, 1996). It was also clear from the findings of this study that students can use non-mathematical interpretations to make sense of unfamiliar fraction situations in the context of problems; It can be seen from Table 1 that Items 12 and 13 were solved with greater success than many of the other items in spite of the fact that they could be classified as division with fractions. Both items were given in the context of a problem. For example, the success on Item 13 was 20%, compared with that on Item 10 which was not given in a context and was solved with less success (8%) by older students. There was also other evidence of students responding sensibly to the contexts provided, for example the student who chose $\frac{3}{5}$ rather than $\frac{3}{4}$ of a pizza (Item 5) "because I don't eat a lot", or the students who said "I would like that because I have no pizza. I would like both (pieces) of fraction pizza" and "No, because is it not a hole but me I want a hole"!

Based on the analysis of our findings in this base-line study, and on the evidence that students can make better sense of unfamiliar situations with fractions in a problem context, we have developed material and a programme for in-service training in our four schools. The approach is problem-centered (e.g. Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991; Olivier, Murray & Human, 1990). Different meanings of fractions and operations with fractions are developed using a rich variety of carefully-selected problems, supported by a learning environment that encourages reflection and social interaction. Teachers do not demonstrate solution methods for problems, but expect students to construct their own strategies, and depend on peer collaboration for error identification and the development of more powerful strategies. Written symbols and the introduction of symbolic algorithms for operations with fractions are delayed until students have had the opportunity to conceptualize fractions as single quantities (Baroody & Hume, 1991; Empson, 1995). There is already some evidence of the success of such an approach with young children (e.g. Empson, 1995; Murray *et al.*, 1996). Our next hypothesis, which we are currently investigating, is that this approach contains at least some elements which directly address the causes of misconceptions discussed in this paper, by encouraging students to construct their own knowledge and by attempting to establish social procedures like discussion and justification to monitor and improve the nature and quality of those constructions.

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Development of the concept of Conservation of Mass, Weight and Volume in children of Mezam Division, Cameroon.

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ABSTRACT

This study investigated the acquisition of the concept of conservation of mass, weight and volume. Conservation in mathematical terms is the invariance of property under some transformation. 120 subjects, 60 from urban and 60 from rural communities who were matched for age, sex and the number of years (3-7) spent in school participated in the study. The instrument used was a modified version of the traditional Piagetian tasks (Piaget and Inhelder 1941; Elkind 1968) of Mass, Weight and Volume in clinical interviews. The results showed no significant difference between boys and girls and a significant difference between rural and urban subjects, with rural subjects conserving better. There was a significant age difference. The "decalage" between conservation of weight and volume was not as distinctive as it is with European studies.

INTRODUCTION

In the wake of independence, many African countries recognized the reality that to ensure a high quality of man-power to support its development, they must invest heavily in their education system. However, they have tended to emphasize quantity at the expense of quality in that educational effort. This has been the case in Cameroon were a very marked progressive increase in enrolment has been experienced in Nursery, Primary, Secondary and Higher Education. In spite of such achievements, the Cameroon school system continue to experience many qualitative problems: shortage of teachers, paucity of instructional materials, prevalence of poor teaching methods, teachers' inability to diagnose students' learning problems, high rate of underachievement, a higher drop-out rate and so forth. The problems of underachievement would be addressed more effectively if teachers and the school system begin to give attention to intellectual developmental processes, to how children learn and the problems they encounter in the learning process. To this end, this study focuses attention on some of the factors that may influence the educational development of the child. Specifically it probes the acquisition of the major concepts of cognitive development:- the concept of conservation which Piaget (1952) considers to be, "a necessary condition for all rational activities".

Theoretical Background

Jean Piaget (1896-1980) worked on cognitive development. His objective was to formulate a genetic epistemology of intelligent behavior. He has provided Psychology, education and other behavioural sciences with an understanding of the active mind adapting to changes in, and pressures of the environment. Piaget believed that the individual tends to organize his behavior and thought to adapt to the environment. These tendencies result in a number of intellectual structures which take different forms at different ages. The child progresses through a series of stages each characterized by different intellectual structures before attaining the age of adult intelligence (Ginsburg and Opper, 1979, 20). Piaget's clinical method of studying these intellectual structures led him to identify four major stages of intellectual development.

- The Sensori-motor stage (from birth -2 years)
- Pre-operational state (2 years - 7 years or 8 years)
- Concrete operational stage (7 or 8 years - 11 or 12 years)
- Formal Operational Stage (11 or 12 years and above)

Although these stages are only rough estimates as they vary from individual to individual, from culture to culture and from one social environment to another, they have been corroborated by other investigators (Pinard & Laurendau, 1969; Lovell, 1959, 1961). The acquisition of the concept of conservation generally occur during the concrete operational stage. For this reason there is a need to examine this stage in detail.

The Stage of Concrete Operations (7/8 - 12 years)

The clearest indication that a child has reached this stage is the presence of conservation - the ability to reverse internally, to decentre, that is, to take into account more than one feature at a time and to focus upon the transformation between one state and another. Piaget frequently found that there is a definite line of Progression in the attainment of conservation with respect to mass, weight and volume. Usually, conservation of mass appears first followed by weight and lastly towards the end of the concrete operations period, volume. Conservation develops gradually and progressively and when it does, the underlying logical structure is the same. These logical structures called the structure of "grouping." The psychological existence of groupings can be inferred from the responses of a child (justification), for without grouping there can be no conservation. If a child is capable of reasoning with the structure of groupings, he knows in advance that the whole will be conserved even though it is broken into parts.

Piaget (1952) identifies five conditions of groupings which form a logico-mathematical model as follows:

i) **Combinativity or closure**
ii) **Reversibility**
iii) **Associativity**
iv) **Identity**
v a) **Tautology**
v b) **Iteration**

When a child has developed the overall structure of groupings, he has available a number of important concepts whose existence marks a considerable advance in logical thought and also enables him to reason in a way not possible in the intuitive stage. **Operational intelligence or operational thinking is fundamentally the application of logical systems in the service of thought**. The child can transform reality by means of internalized actions that are grouped into coherent reversible systems. Studying the development of logic-mathematical thought implies studying the children at the period of concrete operations.

Statement of the Problem

The development of conservation is the focus of this study. The majority of primary school children in Cameroon and in fact all of Africa are in this sensitive childhood stage of mental development. The question then is do these African subjects develop these logical structures in the same manner as European subjects? If there is a variation in these psychological norms internationally accepted, what reason can be advanced to explain this variation. Environment, Age, Sex or what?

Hypotheses

It was hypothesized that there would be no significant difference between the African subjects and Europeans in the acquisition of the conservation of mass, weight and volume.

That urban and rural subjects would not be significantly different in this respect

That Age and sex will not significantly affect the acquisition of conservation of quantity.

Method

The investigator structured interview questions modeled from the original Piaget and Inhelder (1941) conservation tasks. Pliable play dough made from flour using a cook-book recipe was used as substance for the conservation of mass/weight. Conservation of volume was by transformation and displacement. All interviews were conducted by the researcher

I: The test Items for the conservation of Mass (CM) comprised the following:

Two equal balls and three unequal balls were placed before the subject and the subjects were asked the following questions:

1. Choose any two balls out of five balls that are the same. If the subject failed to find the two equal balls, he was asked the next question. (He would choose two unequal balls)
2. What will you do to make the two balls you have in your hands equal? He then proceeds to make the balls equal. After he has ascertained equality of the balls , investigator will proceed.
3. If I make this ball here (take one of two equal balls) into a pancake (flattened shape), will there be the same amount in the pancake as in the ball? (Transformation - mental)
4. I am now making this ball into a pancake (while the subject looks on). Is there the same amount of dough in the pancake as in the ball? (deformation-practical)
5. Why do you think so? (Justification)

II: Test Items for the Conservation of Weight (C.W.). Subjects were asked to weigh two balls (of Play dough) on the weighing scale provided. When they had made certain that the balls weighed the same, the following questions were asked.

1. If I flatten this ball (one of the two equal balls) into a pancake, will the weight of the pancake be the same as the weight of the ball? (the other ball).
2. I am making this ball into a pancake. Does it (pancake) weigh the same as the ball?
3. If I make this ball (the 2nd ball) into a roll (like a fish roll), will the roll weigh the same as the pancake?
4. I am now making the ball into a roll (while the subject watches). Does the roll weigh the same as the pancake? (deformation)
5. Why do you think so? (Justification)

III. Test Items for the Conservation of Volume (C.V.) 2 equal glasses, labelled (A) and (B) contained good drinking water filled to the marked levels, two unequal glasses labelled (C) and (D), (C) tall and slender, (D) stout and fat. Subjects first ascertain that the water in the equal glasses (A) and (B) are equal (same level).

1. The water in glass (A) is poured into glass (C) and investigator shows glass (B) and (C) to subject. "Is there the same amount of water in this glass (C) as in the other (B)?" (Transformation-judgement)
2. The water in glass (B) is then poured into glass (D) (stout/fat), comparing the water in glass (D) to that in glass (A); "Is there the same amount of water in this glass (D) as in the other? (C)" (Judgement)
3. The water was returned to the identical glasses and subject again ascertained that the level of water in both glasses was the same.

   Two equal balls (play dough) were shown to subject. One of the balls was given to subject. "If I put this ball into this glass of water and you put your own ball into the other glass of water, will my ball raise the water in the glass to the same level as your ball? (Displacement-prediction)
4. I am now putting my ball into the water. Can you mark the level of the water. I am making your ball into a pancake (or roll). If I put the pancake into the other glass of water will the water come up to the same level as in this glass? (the glass with ball). (Deformation/Displacement-judgement)
5. Why do you think so? (Justification)

Anticipated Answers and Scoring

1. Conservation of Mass

Correct Responses	Wrong Responses
1. Right judgement – identifies two equal balls (5mks)	Wrong judgement - two unequal balls. (0mk)
2. No task to perform. (5mks)	Wrong response in(1) if corrected in (2) (5mks)
3. Right Prediction-Pancake and ball will be the same quantity. (5mks)	Wrong prediction - inequality of ball and pancake(0mk).
4. Right judgement-amount of dough in pancake is the same as in ball. (5mks).	Wrong judgement - inequality of ball and Pancake (0mks)
5. Right justification (5mks)	No point to justify wrong responses. (0mk)

Note: Justification responses

i Identity - "You did not add any or take any away.

ii. Reversibility: "You can roll the pancake back into a ball and it will be the same."

iii. Compensation: "The pancake is flat but thinner, so it is the same as the ball".

iv. General: "It is the same no matter what shape you make it into, that won't change the amount."

These were the best four responses for which the subject scored 5mks. If he couldn't explain his reason or if he gave a wrong explanation he scored zero (0 mk).

II. Conservation of Weight

Correct Responses	Wrong Responses
1. Right Prediction - pancake will weigh the same as ball. (n.ks)	Wrong Prediction - inequality in weight.(0mk)
2. Right Judgement - The pancake weighs the same as the ball (5mks)	Wrong Judgement inequality in weight. (0mk)
3. Right Prediction - The roll will weigh the same as ball. (5mks)	Wrong Prediction - the roll does not weigh the same as ball (0mk).
4. Right judgement - The roll weighs the same as ball. (5mks)	Wrong judgement they do not weigh the same. (0mk).
5. Right justification as in I above either on the basis of identify, Reversibility, Compensation or general. (5mks).	Whatever reason given is wrong. (0mk)

Conservation of Volume

Correct Response	Wrong Response
1. Right judgement -the amount of water in (c) is the same in (b). (5kms)	Wrong Judgement - Inequality as response. (0mk
2. Right judgement -the water in (d) is the same amount as in (c). 5mks)	Wrong prediction - the level of water will not be the same (0mk).
3. Right prediction - the balls will raise the water to the same level in both glasses. (5mks)	Wrong prediction (0mk)
4) Right judgment - the pancake will raise the water to the same level as the ball. (5mks)	Wrong judgement (0mk)
5) Right justification as in i above, either on the basis of identify, reversibility, compensation or general (5mks)	Whatever reason is giving will be wrong. (0mk)

Note: - Total marks obtainable = 25 marks for each conservation task.

- Any subject who got all responses correct in each section scored 25 marks. Any subject who corrected the response as from item (2) could score a maximum of 20 marks and correction right from item (3) gave a minimum of 15 marks. As from this point even if the response appeared to be correct the subject scores 0 mark.

In the final analysis, the subject scoring 70-75 marks total for conservation were categorized as those having high volume conservation, Those scoring 60-65 marks as high average those scoring 45-55 as average conservation: 60 - 75 – Conservation, 45 - 59 – Transition, 0 - 44 - Non conceivers.

Since the use of verbal explanation (justification) as a sufficient criterion for measuring conservation raises difficulties specially amongst children with less developed verbal abilities, the use of the prediction and judgement tasks (non-verbal) which do not discriminate against subjects who have inadequate language development, were included in the conservation tasks to ensure that the chances of children falling into the category of non-conservers were minimized. Of the 5 tasks in each of the conservations, only one was verbal explanation making it possible for subjects to conserve in the other four tasks. Hence a score of 15 out of 25 or 3/5 was considered good enough to be taken as a conserver in that given task. (Transition)

RESULTS

The analysis of the t-test showed that no significant difference existed between male and female subjects in the overall conservation of mass, weight and volume, and that there is a significant difference between rural and urban subjects. Rural subjects conserved better than urban subjects. Meanwhile the ANOVA of scores on conservation on the basis of Age (3 age groupings - 7/8, 9/10, 11/12 years) showed that there was significant age-group difference. The older subjects performed better than the younger ones in line with European studies. It was observed that for conservation of mass, there was a significant difference between the age groups and a non significant difference between the three groups in the conservation of weight. There was a **very significant** difference between the three groups in the conservation of volume. This difference was shown to be real between 7/8 and 9/10, and 7/8 and 11/12 but not between 9/10 and 11/12 age groups. A Scheffe's test analysis was further performed to confirm the t-test and ANOVA analyzes.

DISCUSSIONS AND RECOMMENDATIONS

Elkind's study (Sigel and Hooper, 1968), which was replicated here, gives mean number of conservation responses for mass, weight and volume, which showed the same order of difficulty as that observed by Piaget (1952) indicating that all subjects scored highest in mass and lowest in volume. This study obtained the following mean scores of conservation responses for the same quantities: mass 14.8, weight 16.5, volume 16.4. These results do not give the same order of difficulty as the order observed by Piaget. The difficulty level is reversed. Children gave less conserving responses in mass than in weight and volume. The fact that the mean scores for Mezam children are higher than those tested by Elkind in the United States should not be surprising since Elkind included 5 - and 6-year-olds who scored less than the older children. It seems for Cameroon children that the decalage between conservation of weight and volume is not as distinctive as it is with the American children. This difference could be attributed to the environmental factors - the subsistence agricultural life that the Cameroonians lead, with much practical and concrete use of materials as opposed to the much more abstract and analytical thinking emphasized by Western Education. In the present study, an f-ratio for age level was observed to be 8.97, ($Pr > F = 0.0002$ and Crit. $F = 3.07$), and very significant at the 0.05 level of significance, which also confirms that conservation responses for Cameroon children increased with age.

Elkinds study showed that differences between age groups appeared as the effect of the type of quantity.

In this study, the multiple comparison of means showed that for mass, weight and volume respectively, the 7-8 and 9-12 groups differed significantly in the number of conservation responses given.

Elkind converted his results into percentages for comparison with Piaget's criterion of 75%. The results of this study were also converted into percentages for comparison with Elkind's results. Percentages are given in Table 1 below. Observation of Elkind's results showed that for mass, the 75% criterion was reached at age 9 (86%); for weight, it was reached at age 10 (89%), and for volume, it was not achieved at age 11 (25%).

Table 1: Percent * of conservation responses for mass, weight and volume at successive age levels (N. 40 at each Age group)

Type of Quantity	Age Levels		
	7-8	9-10	11-12
Mass	45	77	75
Weight	57	72	70
Volume	47	80	85

* of 75 possible scores.

Table 1 shows that the 75% criterion for mass was reached at age 9-10 (77%). For weight it was not reached at all while for volume, it was reached at 9-10 (80%) seemingly earlier than Elkind had found. Thus making weight the most difficult to conserve for all groups.

In conclusion, this study, unlike Elkind's does not confirm Piaget's assignment of the conservation of mass to ages 7-8; the conservation of weight to ages 9-10 and the conservation of volume to ages 11-12.

The Justification Question

American Children:

Children's explanations in Elkind's study were categorized in the following manner:

a. Romancing: "It's more because my Uncle said so"

b. Perceptual: "It is more because it is longer (thinner, wider, narrower, flatter)"

c. Specific: "You did not add any or take away" (identity)
"You can roll it back into a ball, it will be the same" (reversibility)
"The roll is longer but thinner, so it is the same" (Compensation)

d. General: It is the same because no matter what shape you make it won't change the amount".

Categories (a) and (b) were found to be given by non conservers. Elkind reported that romancing and perceptual explanations first increased and then leveled off with age.

Cameroonian Children

Explanations given by the Cameroonian children fitted into three of the four categories given by Elkind's subjects - **perceptual**, **specific** and **general**. The explanation question was the one that posed the most difficulty. Most of the non-conservers gave vague answers which showed their uncertainty. There were a good number of perceptual explanations even in the 9-12 year groups. Specific explanations increased with age. Most of the children who achieved conservation did so through **identity** rather than **reversibility** or **compensation**. All conserving responses involved the previous identity or equality of the two balls.

Finally, of the 120 subjects, 14 reversed the order of conservation difficulty. 18 children conserved weight before mass, 19 conserved volume before weight and 10 conserved volume before weight and mass, 13 subjects did not achieve any of the concept of conservation of mass, weight or volume. While conservation of mass has been reported to be the easiest, in this study, for most of these children it appeared to be difficult. These individual variations must be taken into account and kept in mind when abstracted average statements are made. While there were differences between age groups, it was not possible to predict individual performances according to subject's age, because in some cases young children who were not expected to conserve some quantities **did** conserve, while older ones who were expected to conserve **failed** to conserve.

Educational Implications and Recommendations

The following are some educational implications and recommendations adduced from this study:

1. Children studied here can benefit from Piagetian diagnostic tests and from Piagetian-oriented curricula which has met with success among European children.
2. Individual experiences account for vast variations found among groups and among individuals, - although other factors cannot be left out, such as the subject's personality, the experimenters bias and familiarity with test materials (e.g - the play dough).

a. To what extent do the materials used for a particular tasks and the procedures adopted affect performance? How can the basic competence of the child be determined rather than just his performance in a specific situation.

b. How does performance on Piagetian tasks relate to school performance in other situations? To what extent does either measure indicate desirable out-comes of science and mathematics.

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SUBTRACTION STRATEGIES IN DEVELOPMENTAL PERSPECTIVE: A COMPARISON OF MATHEMATICALLY NORMAL AND MATHEMATICALLY DISABLED CHILDREN

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Abstract

This study was designed to investigate the character and extent of differences between mathematically disabled children (MD children) and their mathematically normal peers (MN children) as reflected in the use of task-specific strategies for solving elementary subtraction problems as children move up through primary school, i.e., from grade 1 to grade 7. The pattern of development showed the MD children as being characteristic of: (1) use of backup strategies only, (2) use of the most primary backup strategies, (3) small degree of variation in the use of strategy variants and, (4) limited degree of change in the use of strategies from year to year throughout the primary school.

Introduction

Many researchers have examined the issue of problem solving in mathematics, and considerable progress has been made during the 1980s and 1990s in describing the problem-solving process. The nature and influence of what affects problem solving has been described from many different perspectives. Among the most critical factors that have been shown to be associated with performance in mathematics includes the varying use of problem-solving strategies. For instance, investigations concerned with development of problem-solving strategies used by mathematically normal children have shown an obvious progression, over time, from immature, inefficient counting strategies, through verbal counting, and finally to automatic fact retrieval from long term memory as children move through primary school . Thus, a normal development reflects an increase in the use of retrieval strategies, and a decrease in the use of backup strategies (e.g., Ashcraft, 1992; Carpenter & Moser, 1984; Geary, 1993; Siegler & Jenkins, 1989). A growing body of research has provided useful information regarding the strategy characteristics of mathematically disabled children (e.g., Geary, 1993; Geary &Burlingham-Dubree,1989; Goldman, Pellegrino & Mertz, 1988; Jordan, Levine

& Huttenlocher, 1995; Siegler, 1988). Previous studies, however, have focused more or less exclusively on single age-groups and on the youngest age-groups in particular. What characterises strategy use, as this develops year by year during the primary school stage, has not been adequately studied.

The present study

The present study was designed to determine potential deficits associated with *the pattern of development* that unfolds when children move up through primary school, as reflected in the use of task-specific strategies in subtraction. The central theoretical viewpoint in the research includes *aspects of strategy variability* as a fundamental characteristic of mathematical cognition. In particular, four aspects of strategy variability were applied through an examination of (1) the use of backup strategies versus retrieval strategies, (2) the use of specific backup variants, (3) the number of different strategies used, and (4) the changes in strategy-use as children moved up through primary school, i.e., from grade 1 to grade 7.

Sample. The sample included 32 MD pupils in grade 1, 33 MD pupils in grade 3, 36 MD pupils in grade 5 and a corresponding number of MN pupils in each of the grades.

Design. The children were observed systematically over a period of two years, grade 1 children from the end of grade 1 to the end of grade 3, grade 3 children from the end of grade 3 to the end of grade 5, and grade 5 children from the end of grade 5 to the end of grade 7.

Procedure. The procedure was almost identical to that followed by Siegler (Siegler, 1988, p. 844). More specifically, the pupils were asked to solve 28 single-digit subtraction problems on two different occasions (T-I and T-II) separated by an interval of two years. The task-specific strategies used by the pupils were recorded on a "trial-by-trial basis" and classified as defined single variants of backupstrategies and retrieval strategies, respectively.

Summary of the findings and discussion

1. Across times of measurement the MN children showed an increased reliance on retrieval strategies, and a decreased reliance on backup strategies. This change in distribution of strategy-use was consistent with earlier research assessing the strategy development of basic arithmetic skills (Ashcraft, 1992; Geary 1993; Goldman et al., 1988; Siegler, 1988). However, from grade 1 to grade 7, no more than 40 per cent of the backup strategies had been replaced by retrieval strategies by MN children. These findings provide substantial support for arguments advanced by Siegler (1987b) that children make use of a mixture of strategies, usually combining counting with direct retrieval. In contrast to the MN children, the MD

children characteristically used backup strategies almost exclusively (close to 100 per cent) throughout the same period. This reflects memory retrieval deficit (Garnett & Fleischner, 1983; Geary, 1992; Goldman et al., 1988). The MD child's typical lack of retrieval activity throughout the whole primary school, as exhibited in this study, can be said to confirm earlier results (Goldman et al., 1988; Ashcraft, 1992). Consistent with developmental difference model (Goldman et al., 1988), the acquisition of strategy skills in subtraction by MD children seemed to follow a sequence that is fundamentally different from that observed in normal achievers.

2. In the case of the MN children, a course of development was observed involving an age-determined shift in strategy-use away from the most primary counting strategies. This implied that other variants, especially verbal counting, were used more frequently (Carpenter & Moser, 1984). It was expected that the corresponding data for the MD children would reflect a developmental delay model, establishing that the difference between the two ability groups would converge early in the elementary school years (Goldman et al., 1988, Geary, 1993). Unexpectedly therefore, the typical MD children continued the use of primary backup strategies throughout the whole primary school stage. These results seem to conflict with the arguments proposed by Geary (1993) that the development of the procedural and memory-retrieval skills of MD children are largely modular; that is, functionally distinct.

3. Usually, the MN child used several different variants of the two main types of strategy (i.e., backup or retrieval strategies). A course of development was observed which showed a gradual but marked increase in the number of strategy variants as the children became older. This result suggests that these children have at their disposal a rich amount of domain-specific strategy-knowledge, i.e., factual knowledge about various strategies and their areas of application. In the case of the MD child, however, the course of development was characterized by far less frequently use of a large number of different task-specific strategies (1-2 variants only) throughout the primary school. This result gives argument to the suggestion that the MD child's insufficient domain-specific strategy knowledge in itself limits the choices available to him/her.

4. The comparison of strategy-use at T-I and T-II showed that the MN children, who had already employed several strategies two years before, continued to change their strategy-use in the direction of new strategy variants (at T-II). The MD children on the other hand, were less likely than MN children to use different strategy variants for the same subtraction problems on repeated testing. The typical MD child seems to use the same strategy variant(s) again and again, year after year, right through the entire primary school. This pattern of development and the similarity between this pattern and the pattern of development showing the number of strategies used, provide substantial support for the suggestion that inefficient strategy-use might be a consequence, in part, of perseverative use of primary

backup strategies (e.g., Goldman et al., 1988).

In all: The pattern of development as reflected in the strategy-use for solving simple subtraction problems applied in a long-term perspective throughout the elementary school years shows the typical MD child as being characteristic of: (1) use of backup strategies only, (2) use of the most primary backup strategies, (3) small degree of variation in the use of strategy variants and, (4) limited degree of change in the use of strategies from year to year throughout the primary school.

Within the whole period covered by the study, the MD children used subtraction strategies which, in a normal course of development, would be typical of the youngest MN children. The use of strategies seems to have been almost permanently established at the end of first grade. The early and striking stagnation shows a strategy development that is divergent from the development observed for the MN children. As indicated above, results from this study reflect a developmental difference model in respect to all (four) variables included.

There is no simple explanation for the pattern in the results. The matter of mathematical difficulties can be explored from many different perspective (Geary, 1993). In particular, two broad perspectives have been suggested to account for the differences between MN children and MD children in the present study, i.e., the quantitative and the qualitative perspective in the strategy-use. In the case of *the quantitative perspective*, attention is focused on the amount of factual knowledge the child possesses about the range of strategies available and their applicability. Research has suggested that domain-specific knowledge, i.e., substantial factual knowledge, is an important component in an effective strategy-use (Pressley, Brokowski & Schneider, 1990; Ohlson & Rees, 1991). Thus, if this suggestion is valid there is reason to assume that the amount of domain-specific strategy knowledge, will be reflected during the problem-solving process through the range of variation in the strategies used (Kolligan & Sternberg, 1987). The characteristic pattern of development for the MN children as they move up through the grades of the primary school, which accentuates that they use a number of different strategy variants to solve the problems in hand, exhibits a richness of strategies, i.e., of substantial strategy knowledge. The fact that in this study the typical MD child seldom varied his/her choice of strategy could be a sign of a lack of strategies. Accordingly, there is evidence that the quantity of domain-specific strategy knowledge might be a critical factor for normal development (Ostad, 1997).

However, the inefficient strategy-use observed among the MD children can hardly be explained as a function of a lack of strategies only, regardless of the quality of the strategy knowledge. According to *the qualitative perspective* in the strategy-use differences can be expressed as differences in metacognitive competence, an expression referring to the pupil's conscious knowledge about when, where, how and why different strategies are effective (Flavell, 1987). When each child's use of strategy at T-I was compared with his/her use at T-II, it was found that the MN children, who had already used several strategies to solve their problems two years before, continued to change their strategy-use in the direction of new strategy variants. The result could be a sign of greater knowledge, but could also relate to the generality of this knowledge indicating that children have the ability to "call forth" appropriate strategies by actively selecting and judging between the strategies at their disposal (Ashcraft, 1992; Brown, 1987). In this case, the children have achieved what can be called *strategic flexibility*. They have improved the quality of their knowledge about the different strategies, giving them greater flexibility in adapting their strategy-use to external and internal (cognitive) variations from one situation to another during the problem-solving process (Ashcraft, 1992; Geary & Burlingham-Dubree, 1989; Siegler & Jenkins, 1989; Schneider, 1993). The MD children, on the other hand, do not change their strategy-use to nearly the same degree as they move up through the grades, so their pattern of development is characterized by *strategic rigidity*. This apparent rigidity or dogmatism among the MD children, could be the result of a limited and functionally inefficient pool of strategies from which to draw upon (Ostad, 1997).

The findings in the present study have educational implications. Special and general educators should bear in mind the suggestion that children can be *channelled into* inappropriate development patterns, for which teaching itself might be partly responsible. The consistency with which some MD children used various backup strategies is an indication that these strategies may very well have been restrictively taught. There are at least two possible explanations for this suggestion, and it is quite possible that both are true. One is that the characteristic pattern of development of the MD children might have been created by excessive emphasis on teaching methods that invite the use of primary counting procedures. This seems particularly relevant when teaching for the youngest age groups (as in the schools included in the study) is based to a large extent on ready-printed exercise books, often with concretes functioning as counting instruments; the main thing the pupil has to do is to count the concretes and write in the answers. The second explanation is that the MD children's strategy-use reflects the fact that nowadays schools spend relatively little time performing mechanical calculations mentally. These children may need not so much to be highly accomplished and accurate at using any particular strategy as to be flexible in applying different strategies in different situations. One could, for example, expect strategy variability to be influenced by the extent to which an individual's school instruction has encouraged or discouraged such variability.

As we have seen, children who have mathematical difficulties require more than ready-printed exercise books, concretes, or real-word practice in solving mathematical problems to become good problem solvers. To address the needs of the MD children, the instructional methods generally needs to change focus, early in elementary school years, from how to learn more mathematics to how to learn mathematics by means of appropriate approaches; that is, providing MD children with instruction to help them to become good strategy users and move beyond rote application of basic skills (Ostad, 1992). These children need explicit instruction in problem-solving strategies as well as guided learning experience in mathematical problem solving (Montague, 1997).

Several projects were initiated to address the need for better mathematics instruction. The importance of metacognition to mathematical problem solving is well acknowledged in the literature (Hiebert & Carpenter, 1992). Cognitive strategy instruction is a promising alternative to current approaches for teaching mathematics to students with learning difficulties. Intervention research focusing on instruction for mathematical problem solving has provided evidence that pupils who lack problem-solving strategies need explicit instruction to faciliate their reading, understanding, executing, and evaluating of problems. In contrast, students who have a repertoire of problem-solving strategies but use them inefficiency or ineffectively may need metacognitive strategies (e.g., self-instruction, self-monitoring, self-evaluation) to help them activate, select, and monitor strategy-use (Graham & Harris, 1994). However, whether MD children would become good strategy users after extended strategy instruction in the early age groups remains an open question. More definitive answers to this question must await further research.

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REPRESENTING PROBLEMS IN THE FIRST YEAR OF SCHOOL

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This paper describes the development of representations of numerical problems by children in their first year of school. Children's drawings of problem solutions were obtained from two classes over the period May to November in successive years. Both classes were taught by the same teacher. The children solved a range of numerical problems by modelling them with concrete materials, then drawing representations of the problem situations. Over the period of the study the representations became more structured, with groups delineated in various ways, including use of letters and labels. By the end of the year, the children were writing their own problems and many children were also representing problems symbolically.

INTRODUCTION

Problem solving is an umbrella term that subsumes many cognitive processes, including among others, establishing goals, verbal and syntactic processing, change of representation, and algorithmic processing (Goldin, 1992). Representation is an important area of problem solving and little is known about the relationship between internal understandings and representations and their external counterparts (Mariotti & Pesci, 1992). For example, how do young children learn to represent problems and to make connections among the different representations (concrete, pictorial, symbolic and mental) generated as part of the problem solving process?

The relation between internal and external representations has been explored by Cobb, Yackel and Wood (1992) who believe that mathematical meanings given to representations result from interpretation and construction by students. The idea that children develop individual interpretations of concepts, has changed the emphasis in mathematics teaching, from producing answers to the solution process, including how children interpret problems, as well as how they represent and explain their solutions. While representational systems may be individual, Thomas and Mulligan (1996) suggest that the further the representational system has developed structurally, the more coherent and well organised will be the external representations that students use. But what are signs that indicate increasing structural development?

When one structure is represented by another, a crucial understanding is the extent to which the original structure is preserved in the representation (Kaput, Luke, Poholsky and Sayer, 1987). Two factors have been suggested as important in representing problem structures: incorporation of numerical information from the problem into the representation and clear depiction of the relationships among problem quantities (Lopez-Real and Vello, 1993). These factors are essential features of representing numerical problems, however, both perceiving and representing such features may be difficult for children. Recognition of such features when a representation is provided is difficult. Beveridge and Parkins (1987) found that provision of a diagram or a model indicating the procedure required to solve the problem was only effective in helping students solve it when the representation was such that students recognised the structural correspondence between the diagram or model and the problem. The situation appears even more complex when children create their own representations, especially in the beginning years of school when such representations may be difficult for teachers to interpret and for children to explain.

Concrete materials have been suggested as a way young children can model solution processes because such representations mirror conceptual structures. Therefore, the structure of the representation may assist children to construct a mental model of the concept, providing children see the correspondence between the structure of the material and the structure of the concept. An inability to recognise such structural similarities has been suggested as a reason why concrete representations do not always assist children to learn about particular concepts (Dufour-Janvier, Bednarz, & Belanger, 1987; Hart, 1987; Janvier, 1987). Similar concerns have been expressed about visual representations. Dubinsky (1989) commented "It may be true that a picture is worth a thousand words, but what if it is the wrong picture?" A child's representation may have omitted key information or coded in a form that is difficult to use (Davis, 1984). Another concern is that children may not be taught to represent problems as a means of finding solutions. In a study of Year 5 and 6 only 5% drew diagrams for 693 "diagram-suitable" problems (Lopez-Real and Vello, 1993). When the children were asked to draw diagrams and use these to solve problems, correct solutions were given to approximately one third of problems previously answered incorrectly.

If a representation is provided students may not recognise structural similarities between a situation and its representation but for students to create their own representation requires knowledge of the conceptual structure and articulation of its essential features. So how do students learn to recognise structural features and incorporate these into their mental models? One important factor in recognising and representing structural features would seem to be experience of translations among the different types of representations, as well as translations within each. This factor has been emphasised by Lesh, Landau, and Hamilton (1983). These authors consider that the act of representation may facilitate the emergence of concepts and representations during problem-solving sessions as students use different representation systems, in series or in parallel, to solve problems. They found that in problem-solving situations, good problem-solvers were usually able to switch to the most convenient representation during the solution process.

In a synthesis of the literature on problem solving Lester (1996, p. 666) suggests that to become successful problem solvers, students need to solve many problems over a prolonged period of time. Moreover, most students benefit from planned instruction, although teaching them about problem solving strategies does little to improve their ability to solve mathematics problems in general. He also commented that for students to benefit from instruction, they must believe that their teacher thinks problem solving is important. The first point would suggest that problem solving should start as early as possible in a child's schooling. Indeed it has been shown that kindergarten children could solve multiplication and division problems when they were encouraged to represent relationships or actions described in the problems (Carpenter, Ansell, Franke, Fennema, and Weisbeck 1993). These authors felt that assisting children to consolidate and extend the intuitive modelling skills they apply to problems might provide a framework for developing problem solving in the primary school.

The other points Lester (1996) makes stress the importance of the teacher's role in assisting students to value problem solving as well as to solve problems. The present study is an intensive investigation of how children's representations of numerical word problems develop in the first year of school when the relationships among different representations (concrete, pictorial, and symbolic) and emphasised and how children's understandings of such relationships change over time.

METHODOLOGY

The children in this study were all from 5 to 6 years of age and were enrolled in kindergarten in a school in a medium socio-economic area of Sydney. They were separated into four groups early in May 1996, although these groupings were not fixed and some children later changed groups. From May until November each group worked on problem solving for 30 minutes once a week during the mathematics activity time. The groups rotated through different activities over the week; the teacher worked with the problem solving group while volunteer mothers worked with the other three groups. A similar structure was followed in 1997.

The data for this paper is primarily based on the drawings of nine children from the 1996 class whose work was available because they used dark pencils when drawing, thus enabling their work to be photocopied. The drawings were not initially collected for the purpose of analysing their representations but to use as worksamples for student teachers. Only later were the drawings examined with the aim of examining the changes in the children's representations over time. In 1997, the children were observed and taped during problem solving sessions and all their drawings collected but these data are yet to be analysed in depth.

The contexts of the problems were based on either literature read in class or familiar situations. In the problem solving sessions the teacher began by reading the problem to the children and prompting them to model it with cubes. This step was crucial because the children could not read the problem so the cubes acted as a memory aide. The children were then asked to draw a picture to show how they solved the problem. The groups generally attempted the same problems although some children attempted additional problems and some children required far more support than others. In later sessions the children progressed to writing their own problems. The problem types (Carpenter et al., 1993) included addition, subtraction (combine and separate), multiplication (equal groups), division (partition and quoitition).

RESULTS

In both years the children's drawings developed markedly over the two terms. None of the children's initial drawings were structured but the later drawings of all children showed evidence of their solution strategies and most children's representations were increasingly organised in the way both quantities and relationships were depicted. Because the cubes that the children used to model the problems were coloured, it would be expected that colour would be used as a means of depicting problem parameters and relationships and this was so. However, the children developed other drawing strategies including:

- size (eg adults, children) and key features (not detailed drawings);
- separation for subtraction and addition;
- crossing out and partitioning of sets for subtraction;
- drawing lines to indicate sharing relationships;
- array structure to show equal groups in a multiplicative situation;
- letters and words to label elements of sets or sets.

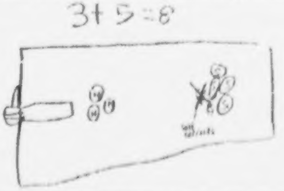
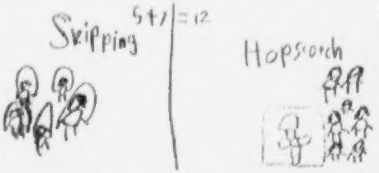
Eight of the children at the party are trying not to get sunburned noses. Five are wearing suncream and the rest are wearing hats. How many children are wearing hats?"	There were 12 children in the playground. Seven children were playing hopscotch and the rest were skipping. How many children were skipping?
	
Jeffrey - 13/8/96	Jeffrey - 3/9/96

Figure 1 Jeffrey's labelling of groups

While the use of size, pictorial details, and separation to show groupings would seem to be a natural part of drawing a picture for young children, use of letters to label group elements or words to label the groups might not be predicted. Jeffrey (see Figure 1) suggested using letters to label the elements of a set, e.g., H for hats and S for sunburned. In a subsequent problem Jeffrey labelled groups, rather than individual elements. The impetus for this strategy may have come from the emphasis on initial sounds in early reading. Ideas such as this were often adopted by other children in the group.

Figure 2 shows the structures used by some of the children. Joel has used a very clear method for showing that two children from one team had to move and has also written a number sentence to represent this. The other two problems in Figure 1 involve grouping, Edward counted out 15 cubes, then separated these into groups of three. In his drawing he has used lines to connect each group and written a repeated addition number sentence. Shani copied the correct structure of a wall made of Lego blocks and wrote a number sentence to represent this structure. Late in the year calculators were introduced to the problem solving sessions and children used them to solve their number sentences and also to experiment (see Jake's example).

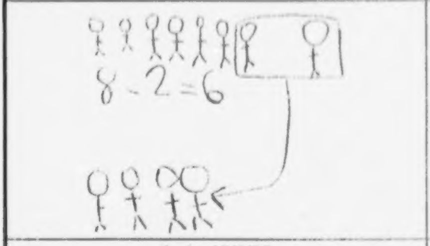
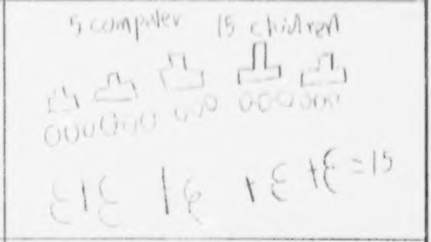
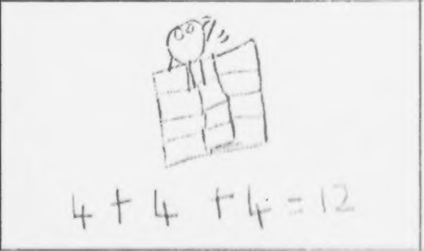
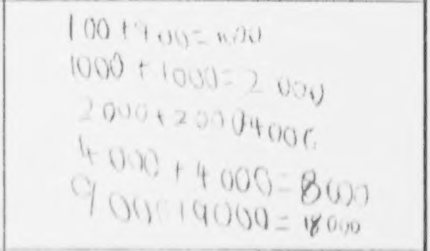
It is time for gym and the children have to make 2 teams. One team has 8 children and the other has 4 children. How many children have to move to make the teams equal.	Today KS are using the computers. If there are 15 children and 3 children can share each computer, how many computers will we need?
	
Joel - 13/8/97	Edward - 19/11/97
Here is Humpty Dumpty sitting on his wall. Draw Humpty sitting on his wall and write a number sentence about the wall.	Write the most interesting number sentence you can think of. Use your own way to work out the answer.
	 100+900=1000 1000+1000=2000 2000+2000=4000 4000+4000=8000 9000+9000=18000
Shani - 26/11/97	Jake - 19/11/97

Figure 2 Structures and number sentences from the 1997 data

Use of number sentences

The impetus to use number sentences came from children, in each year a child asked about the "plus" sign. The teacher worked through an example with the group and assisted children to write number sentences. In the 1996 group, six of the nine children did not seem to have difficulty writing both addition and subtraction number sentences when these could be clearly related to the children's representations of problem structure. The children who wrote number sentences tended to draw groups that were delineated in some way; that is, separated, labelled, or shown with different pictorial details (such as stick figures wearing hats).

Because the children were translating from a written problem to a concrete, then a pictorial and finally to a symbolic form, production of a number sentence was a mapping of the concrete or pictorial representation rather than of the written problem. Thus, it is not clear from the 1996 data whether the children could have solved the number sentences if there was no accompanying representation (cubes or drawing). In general the children used a count-all strategy to determine the answers to the problems, so most of them relied on an intermediate representation such as the cubes or drawing to work out an answer. As they gained counting skills such representations might not be necessary. The 1997 data indicates that by the end of the year some of the children could translate simple addition and subtraction word problems directly into number sentences and solve these using calculators as well as to begin to experiment with large numbers (See Figure 2).

Self-generated problems

In 1996, soon after the children began writing number sentences the teacher asked them to write their own word problem. In general, the structure of the self-generated problems was modelled on the structure of the immediately preceding problem. However, there were exceptions and one of the later self-generated problems, "Write your own problem about the zoo.", produced more varied responses. The given problem was "Geoffrey went to the Zoo to see all the animals. He brought along ten slices of bread for the elephants. When he had finished feeding them he had two slices of bread left. How many pieces of bread did the elephants eat?" The two children dictated addition problems of the form:

> There was a giraffe, a tiger and a monkey. How many animals were there?

Four other children modelled the same operation as the original problem (subtraction) but the structure of their problem was simpler; instead of modelling a change situation as in the given problem (10-x = 2) they made the result the unknown, for example:

> There was one lion and ten fairy penguins. The lion ate two of the penguins. How many were left? (Craig)
>
> There were five kookaburras and one snake. The snake ate two kookaburras. How many were left? (Anthony)

These children included three parameters in their problems but not in earlier separate situations because most children generated take away situations (birds flying away, koalas climbing down trees, etc.). For the given zoo problem Craig wrote 10-2= , then crossed this out and wrote 10-8=2. For his own problem about the zoo (see above) he confidently wrote 10-2=8. Anthony first wrote 2-3=3, then crossed this out and wrote 5-2=3. The three remaining children wrote their own problems and these were more complex:

> There wor 5 lin and there wor 8 tigrs tace awae too of them ho mach wor there left (David)
>
> There was 10 stars and 15 children ho memey mor stars do we need? (Alison)

> Threr worer five benarnas and 2 muncies. bothe of the 2 muncies aete 1 eche. How meny benarnas were left? (Jeffrey)

None the above structures had been given to the children: David and Jeffrey both developed multistep problems while Alison wrote a compare subtraction structure. Alison and David attempted to write the number sentence; David wrote 5-8=12 while Alison first wrote 10+15, then crossed it out and wrote 10-15=. The reason for the increasing complexity of these problems may have been that the preceding problems were not simple structures to model but ones in which the change had to be determined given initial and final states.

CONCLUSIONS

Although the sample is not a representative one, the results for these children support those of Carpenter et al. (1993) who showed that after eight months in kindergarten children who were taught problem solving could solve a variety of quite difficult word problems. The children in this study had relatively little time spent on problem solving, yet they were remarkably successful in representing and solving complex word problems. The children's drawings of the problem situation show that they used a variety of strategies to represent aspects of the contexts including showing properties of the problem elements (colour, size, pictorial details); separating groups or crossing out individual elements; partitioning sets and drawing lines to indicate sharing relationships; drawing array structures to show equal groups in a multiplicative situation; and using letters and words to label elements of sets or sets.

All nine children had written number sentences for single step problems with a direct relation between the quantities by the end of the year, and some had a good grasp of representing problems symbolically. A preliminary analysis of the data from 1997 suggest that the children had gained more control of linking concrete, drawn and symbolic representations than the 1996 group, perhaps because in 1997 began writing symbolic expressions and writing their own problems earlier than the 1996 group.

The results presented in this paper suggest that in the beginning stages of problem solving children can make considerable progress when:

- children model situations with concrete materials but are also required to draw a representation to show their thinking;
- children are expected to explain their solutions and their attention is drawn to the connections among representations;
- language structures are emphasised and problems are linked to familiar contexts;
- problem solving is seen as an enjoyable activity valued by the teacher;
- the teacher has an expectation that all children will benefit from problem-solving with scaffolding provided to support children at different levels; and
- problem solving skills are consolidated and extended over a period of time.

The value of linking modelled and drawn representations may be that the drawn representation gives children a means of recording their thinking and using this to explain the solution process. Moreover, asking children to explain how their drawings and their models are the same focuses their attention on similarities between the two structures from the beginning. If the same strategy is followed when number sentences are introduced, children develop number sentences that directly relate to their solution strategies. If the above conditions are part of the teaching environment, then some children achieve far more than would be expected at kindergarten, both in terms of representing problem situations symbolically, and in generating problems that they have not previously encountered. The children also enjoyed problem solving and this initial enthusiasm needs to be fostered in subsequent years.

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OPERATIONAL AND CONCEPTUAL ABILITIES IN THE LEARNING OF ALGEBRAIC LANGUAGE. A CASE STUDY

PALAREA, M.M. and SOCAS, M.M.

We present a student's biographical case study by means of the data obtained from various instruments (pre and post-test, classroom notebook and interviews). The study provides us with information about the student's conceptual and operational cognitive processes when learning algebraic expressions. This work forms part of a much wider project in the design of a Curricular Proposal for algebraic language in the Spanish Secondary Education system. Our Curricular Proposal takes into account the operational and conceptual cognitive abilities that facilitate transition from arithmetic to algebra, based on the four sources of meaning and within the framework of Duval's work (Semiosis-Noesis).

Over the past twenty years there has been enormous interest among both researchers and teachers in the study of the difficulties involved in the teaching-learning of school algebra. However, the problems in this area have not been resolved and what should be taught and learnt in algebra has yet to be decided.

As Kieran (1992) points out, algebra implies: the recognition and use of structures, the meaning of the letters in the algebraic context, and the change to a series of different conventions other than those used in arithmetic. As such, a procedural approach, based solely on a simple generalization of arithmetic, does not seem suitable.

Sfard (1991) shows that abstract notions in mathematics, such as algebraic language, can be basically conceived of in two different ways: structurally (as objects) or operationally (as processes). The existence of historical stages during which the different mathematical concepts, such as number and function, have evolved from the operational to the structural, has led Sfard to design a three-phase parallel conceptual model: interiorization, condensation and reification.

We believe that in the way they are currently presented there is a dilemma regarding the status of the objects of algebra, as well as other mathematical objects: a dynamic operational status whereby objects are seen as a process, and a static structural status whereby objects are seen as a conceptual entity. However, while the structural status of the mathematical object presented is organized into various conceptual networks and is completely accepted, the semiotic representation systems (SRS) characterizing the operational status (numerical representations, algebraic codes, graphs, diagrams, etc.) in which algebraic objects are expressed and by which they are communicated, have received less attention from mathematicians and the educational community.

Kaput (1987) points out that all SRSs possess at least four sources of meaning: translations (conversions) between formal SRSs, translations (conversions) between formal and non-formal SRSs, transformations and operations within a single SRS, independent of any other SRS, and consolidation through the construction of mental objects by means of actions, procedures and concepts present in the intermediate SRS,

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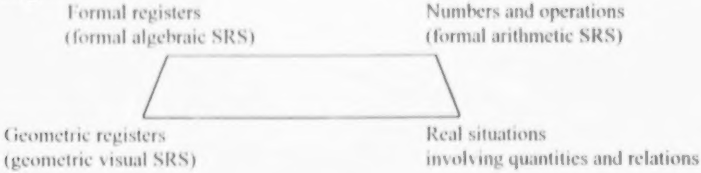
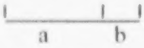
which are created throughout the teaching sequence.

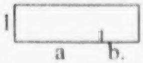
Research into visualization in mathematics and the role played by mental images has shown the importance of representations in the proper formation of concepts. Various researchers -Janvier (1987), Hiebert (1988), Kaput (1987, 1991), Duval (1993, 1995)- have carried out experiments and advanced theory in order to shed light on the articulation mechanisms present within the process of understanding knowledge.

Of all these works we would like to single out Duval's (1993, 1995). Duval performs a coherent task that brings together different theoretical approaches to representations (Semiosis - conception or production of semiotic representation, and Noesis - articulation of several semiotic representations).

From our experimental studies of algebraic language (Palarea and Socas, 1994a, 1994b) we have confirmed the necessity to extend the sources of algebraic language meaning to SRSs derived from visual sources (geometric registers - Palarea and Socas, 1994b). Four sources of algebraic meaning, based on Kaput's terminology, are thus given as follows:



Geometric registers make use of two-dimensional geometric representation to represent algebraic expressions and terms. We propose a Visual Geometric S.R.S. whereby any expression codified in it can be used for both syntactic and semantic purposes. This system is based on operations with magnitudes and it is a system for culturally, didactically and mathematically meaningful representation. Algebraic expressions, then, are contextualized in a system of geometric representation where numbers and letters are lengths of known and unknown segments. Letters are considered either as geometric objects or as generalized numbers, with two different registers: "a" and "1 . a". So "a + b" accepts the registers 

and 

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Calculations in a Visual Geometric S.R.S. are built upon the five properties that characterize the number system: $a + b = b + a$; $(a + b) + c = a + (b + c)$; $a.b = b.a$; $(a.b).c = a.(b.c)$; $a.(b + c) = a.b + a.c$.

In terms of Duval's thesis, the Visual Geometric SRS interacts with the Formal Algebraic Semiotic representation system in the following way:

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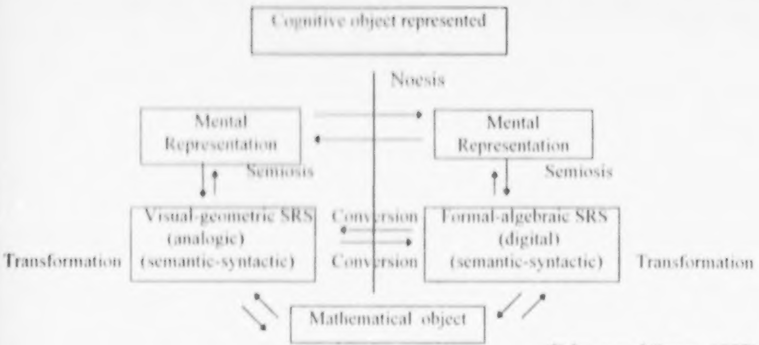
Cognitive object represented
Noesis
Mental Representation
Mental Representation
Semiosis
Semiosis
Visual-geometric SRS (analogic) (semantic-syntactic)
Conversion
Conversion
Formal-algebraic SRS (digital) (semantic-syntactic)
Transformation
Transformation
Mathematical object

(Palarea and Socas, 1997)

The purpose of the present work is to provide information about the operational and conceptual cognitive processes for algebraic expressions manifested in the transition from arithmetic to algebra. We give a biographical case study in which we examine the data obtained from various measurement instruments, including Pre-test and video recordings, together with analysis of students' workbooks, recordings and Post-test.

Method

The student chosen for the case study belongs to an experimental group of thirty-one 12-13 year-old students (7th Primary Grade, Tenerife, Spain). All students underwent a Pre-test and Post-test which were organized around 28 questions related to the operational and conceptual abilities. The students worked on an instruction model designed by the researcher herself. Nine students were interviewed individually in semi-structured interviews regarding the cognitive abilities.

The case-study student was selected on the basis of being an average student within the group as regards academic performance and because the student's observable behavior and reactions (classroom, clinical interview) afford the researcher understanding and explanation of situations of interest.

Tasks and Results

Students' tasks involving operational cognitive abilities are proposed and worked on in a single formal register (Semiosis), and their tasks involving conceptual abilities are proposed and worked on with two registers (Semiosis-Noesis). The results of the chosen are expressed here in their formal registers, except for register conversion tasks. They are denoted as Pe (Pre-test), Po (Post - test), I (Interview) and T (classroom work).

asks	Results
O.1 Carry out arithmetical operations in general or with letters without using brackets	
) The question a + 3 a can be written in a more implified form as 4a. Write the following in a nore simplified way, reducing the expressions to	

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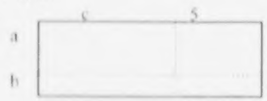
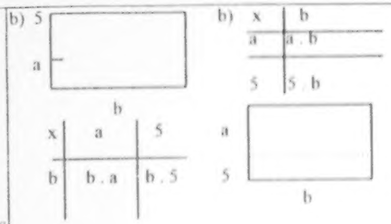
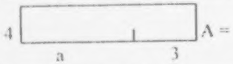
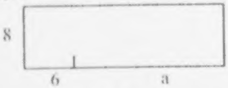
the shortest form possible: a.1) 2 a + 5 a a.2) 2 a + 5 b a.3) 2 a + 5 b + a a.4) a + 4 + a - 4	*a.1) 7 a (Pe. and I); 7 a^2 (Po.).* *a.2) 7ab (Pe and Po.); 7ab, he* express: *"as it is"* and he writes *"7ab" (I).* *a.3) 8 a^2 b (Pe. and Po); 8 a b (I).* *a.4) Not solved it (Pe); 2a^2 (Po); 5 a + 3 a = 2 a (I).*
O.2 Carry out operations with brackets in additive and multiplicative contexts, paying special attention to what is denoted by the brackets	
a) The question a + 3 a can be written in a more simplified form as 4a. Write the following in a more simplified way, reducing the expressions to the shortest form possible a.1) (a + b) + a a.2) (a - b) + b a.3) 3 a - (b + a) a.4) (a + b) + (a - b)	*a.1) 3 a^2 b(Pe. y Po.); 2ab+a = 3ab (I).* *a.2) Not solved it (Pe); 1ab^2(Po.) a b+ b=2b+a (I).* *a.3) Not solved it (Pe). 1b (Po.); 3 a-2 a b=1 a b (I).* *a.4) Not solved it (Pe); 2a^2b^2 (Po.) ; 2 a b + a b = 3 a b (I).*
O.3 Make formal substitutions with reference to both specifying and generalizing processes	
a) Fill in the gaps as shown in the first section of each column: x → x + 4 6 → r → b +2 → Now, if a = 2b, what can 5a + 3 be transformed into?	*Pre-test Post-test Interview* *x → x + 4 x+ 4 x + 4* *6 → x + 6 6 + 4 6 + 4 = 10* *r → r + r r + 4 r + 4* *b +2 → 4 +b + 2 b + 2 + 4 = b + 6;* *10 + 3 = 13 5 a + 3* *5 . (2b) + 3 =* $\boxed{10b+3}$ *a = 2b*
a) Fill in the gaps as shown in the first section of each column: x → x . 4 6 → r → b +2 →	*a)* *Pre-test Post-test Interview* *x → x . 4 x . 4 x . 4* *6 → 6 x 4 . 6 6 . 4 = 24* *r → r^2 4 . r 6 . r = 6r* *b +2 → 4 (b + 2) 4 . b + 4 . 2 = 4 . b + 8*
C.1 Carry out conversions between the various registers, paying special attention to the designations of the brackets in the formal register	
a) The product (a + b) (c + 5) can be written using the area of the rectangle with sides a + b y c + 5, as:  (a + b) (c + 5) = a.c + a.5+b.c+b.5 Write the following products: a) a . (b + 5) = b) (a + 3) (b + 2) = c) (a + b) (a + b) =	*Pre-test* *a) a . b + a . 5* b) *a.b+a.2+3.b+3.2=* *=2a+3b+6* c) *a.a+a.b+ b.a+ b.b=* *=a^2+ab+ ba+b^2* *=a^2+ a b^2 + b^2* *Interview* *a) a . b +a . 5* *b) a.b+ a.2+3.b+3.2* *c) a.a+a.b+b.a+b.b=* *= a^2+ab+ ba+b^2* *= a^2 + 2 $(ab)^2$ +b^2*

<table>
<tr><td>b) Represent (a + 5).b by means of rectangles and use a double-entry chart.

c) Write the following operations with algebraic language:
c.1) Triple of x.
c.2. Double of n minus 4.
c.3) Triple of the addition a and b.
c.4) Double of the difference between h and i.
c.5) Triple of the square of b.</td><td>

b) 5
a
b
b) x | b
a | a . b
5 | 5 . b
x | a | 5
b | b . a | b . 5
a
5
b

c.1) 3x (T, Po and I)
c.2) 2n - 4 (Pe, T and I)
c.3) 3 (a + b) (T. and I)
c.4) 2 (h-i) (T. and I)
c.5) 3 b^2 (T. and I).</td></tr>
<tr><td colspan="2">C.2 Contextualize algebraic language in general, and letters in particular, as geometric objects and as generalized numbers in area and perimeter contexts</td></tr>
<tr><td>a) A school bus driver made "n" trips in a day, carrying 50 children on each trip. How would you express the total number of children he carried that day?.
b) Calculate the area of the following diagram:

c) A newspapers sells 135 newspaper a day. How would you express the total number of newspapers sold in two weeks (14 days)?.
d) Calculate the perimeter of the following diagram:

perimeter =</td><td>a) n= trips
n . 50 = x children (Pe)

b) A= b . a = 3a . 4 =12 a^2 (Pe)

A = b . c = 3a . 4 = 12 a (Po)

c) x = 135 newspapers
d = 14 days
x . d

d) 8 + 8 + 6 + 6 + a + a = 28 a^2
16 12 a^2 (Pe)

d) 8 + 8 + 6+6+ a+ a = 16 + 12
28 + 2 a (I).</td></tr>
<tr><td colspan="2">C.3 Interpret and understand the meaning of algebraic expressions, letters and signs, paying special attention to the use of the equal sign.</td></tr>
<tr><td>Write the following expressions:
n + 1, n + 4, n - 3, n, n - 7. Which is the greatest and which the smallest?.
Justify your answer......................</td><td>(Pe) Not solved it.
(I): smallest n-7 greatest n+4
n + 1, n + 4, n - 3, n, n - 7
8+1=9 8+4=12 8-3=5 8 1</td></tr>
</table>

Discussion

With regard to operational cognitive abilities O.1 and O.2, it can be seen that this student's greatest difficulty is a lack of operational ability with letters, as shown in the examples given. especially regarding addition.

Later this translates into errors when calculating areas where the student tries to express square units by adding an exponent 2 to the final letter. So, the area of 3 b the student expresses this by $3b^2$.

The student evidently assumes that area has two dimensions and is not concerned where these dimensions come from.

Here is a literal transcription of Question a of Item O.1 (S = Student, I = Interviewer)

S: (studying the text and pointing at the "a" in "a + 3a").*Here it's as if it was 1.*

I: *In other words, you think that this is as if you had a 1, don't you?*

S: *Yes*, (he begins to count) 1 *a + 3 a;* ***2 a + 5 a = 7; 2 a + 5 b =****, here you couldn't because they aren't similar terms, you couldn't add.*

I: *OK, then, how would you do it?*

S: *You couldn't add. How can I add it?*

I: *And how does it finish, then?*

S: *Well, like this (2a + 5b).*

I: *Yes, like that, that's it.*

S: *7ab.*

I: *Is that the same? If you think you cannot add you leave it as it is, and if you think that you have to put 7ab, well then ...*

S: *Well, they aren't similar terms, that's what I've been taught, and because they aren't similar you can't add ...*

I: *What's the result, then? As you've got it?*

S: *Yes.*

I: *Well, do it, if you think that it's like that because they aren't similar ...*

S: *No, like this 2 a + 5 b = 7 a b.*

I: *If you think ...*

S: *7ab.*

I: *Let's go on to the third example.*

S: ***2 a + 5 b + a = 8*** a^2 ***b.***

I: *In other words, when you have a + a, it's* a^2.

S: *No a + a = 2a.*

I: *Go on. As far as you're concerned, is this (pointing at 2 a + 5a) and this (pointing at 2a + 5b + a) the same?*

S: *Yes, they're the same.*

I: *They're the same?*

S: *No, the result we'd have for this (pointing at 2 a + 5 b + a) is greater; a + a is 2a squared.*

I: *And this 2a you have here?*

S: *This is multiplied by "a". So, it isn't 8ab squared.*

S: ***a + 4 + a - 4.***

I: *Finally, what do you think?*

S: *Well, here I can't put 4a because that would be multiplying.*

I: *So?*

S: *As I can't put ... here you have a 1. 5a, no. It would be 5 a. This 3a is the same as 2a.*

I: *In other words, you've got this 5. Where ..?*

S: *From the a of a + 4 and the 3 from the 1of the other a minus the other 4.*

I: *I see*

We can see two erroneous expressions used systematically: "2ab" is considered to be the result of "a+b", and in other examples, which we have not published in this work for lack of space, "ab" is considered to be the result of "a-b". Also, there is a tendency to add the coefficients of all the terms, whether or not they are similar. However, in the interview the student states that if the terms are not similar you cannot carry out operations. This type of situation is very common in teaching: the student has the conceptual ability but he cannot transform this into operational ability.

The student's understanding of distributive properties as such is obvious, but his lack of operational ability when adding leads him to make mistakes in the subsequent sequence of operations.

With regard to substitutions (Category O.3) the student is secure and always expresses the product first with an "." and then, making use of the conventions of algebra, he copies the expressions by leaving it out.

The student has no problems in converting representations (Category C.1). He knows the key words used perfectly: triple, double, following, preceding, square, product, difference. His replies are always right when it comes to conversion in context.

A surprising fact is that when carrying out conversions from ordinary language to algebraic language he sometimes fails to make such conversions correctly when the language is closely related to contexts familiar to a student of his age.

When contextualizing algebraic language (Category C.2) the student's lack of operational ability, referred to above, is even more notable when performing calculations in which the letters refer to geometrical objects.

The idea of product associated with the notion of area leads the student to express $3.b = 3b^2$, where the b^2 is an expression associated with area as the product of two dimensions. This is repeated in calculations concerning perimeters, where although it was checked that he had a clear idea about perimeters, he makes mistakes in those operations involving the calculation of perimeters when the diagram (whether it be a rectangle or not) has some of its sides divided in dimension. In such a case, the student performs invalid operations such as "2 + x" for "2x" and "3 + y" for "3y".

The student's conceptual ability regarding area and perimeters has been enriched through a change in procedure, in as much as he does not assign to areas and perimeters units of measurement which are not explicit in the dimensions of the diagrams.

With regard to the interpretation and understanding of the meaning of the algebraic expressions, signs, and letters (C.3) in the question given above in the Table, which proves difficult for most students, this student reasons his replies very well, and does so correctly from the start, without ever giving any numerical value to "n", and then giving "n" the value of 8.

The student understands that when adding a greater number to a given number the result is always greater and that, on the other hand, when subtracting a greater number from a given number, the result of the expression is always smaller.

Conclusions

Among the most important conclusions, the following are noteworthy:

Direct conversion of real situations involving quantities and relations in different contexts does not appear to pose any difficulties when the student knows the key words that permit such conversion: triple, double, following, preceding, square, product, difference, etc.

Conversions between the Visual Geometric Semiotic Representation System and formal algebraic SRS pose difficulties in both area and perimeter contexts, especially when some of the dimensions are subdivided.

Difficulties in operational abilities are more apparent when carrying out additive operations. When performing operations involving the calculation of perimeters, the student makes mistakes even though he has a clear idea about the idea of perimeters. This error is more common when one of the dimensions is subdivided.(Küchemann, 1981; Chalou and Herscovics, 1988).

With regard to problems of multiplication, especially those involving distributive property, the student is sure of himself, both on the left as well as on the right. The subsequent additive situations do not cause any difficulties.

It seems surprising that even when he knows the rule "if two terms are not alike, operations cannot be carried out" the student does not have much additive operational ability. This situation is very common in teaching: the student seems to possess the conceptual knowledge but is unable to transform this into an operational ability. As Booth (1984) states: "the ability to describe a method verbally does not necessarily mean the ability to recognize the correct symbolisation of this method."

Fixed rules outweigh conceptual knowledge. In spite of having incorporated conceptual knowledge of the new register, the underlying operational abilities (formal arithmetic SRS) make the student commit errors because of a lack of meaning in the new register. This lack of meaning is not made up for by transformations in the formal algebraic SRS (semiosis), but is acquired in noetic processes.

The case study student makes suitable articulations between visual geometric SRS and formal algebraic SRS in multiplicative questions, especially when involving distributive property. However, this mental scheme leads him to commit errors in additive contexts when he needs to express the letters as square powers, an expression associated with area.

Based upon the data obtained, we can affirm that the proposed model can be deemed a good method to go deeper into qualitative analyses of the difficulties, obstacles and errors that arise when making the transition from arithmetic to algebra.

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WIDENING THE INTERPRETATION OF VAN HIELE'S LEVELS 2 AND 3

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Despite controversy, the van Hiele levels continue to be used as an important framework for interpreting students' understanding of geometrical ideas. However, as they stand, the level descriptors offer a restrictive base. These cause problems when questions are posed outside of the direct notions of properties of figures, class inclusion and deduction about which the theory is explicit. This paper builds upon an initial attempt to broaden the level descriptors in a way that is consistent with the original ideas of the theory but which allows for more inclusive criteria. This paper reports on three issues. First, examples of the restrictive nature of the van Hiele level descriptions are identified. Second, the relevance of certain notions, drawn from the SOLO model, are explained. Finally, three brief case studies are reported using the new descriptions to provide empirical evidence.

Introduction

The van Hiele Theory has been the focus of considerable research attention, and many theses, reports and articles have been published concerning issues related to verification and exploration of the five hierarchical levels, and their associated characteristics. The levels have proved a useful tool in (i) identifying problems in students' understanding of certain geometrical concepts, (ii) evaluating the structure or development of geometric content in secondary text books, and (iii) guiding the development of syllabuses. However, the theory has not been without controversy. Some (Gutiérrez *et al*, 1991) have challenged the discontinuous nature of the levels. Others have queried the apparently simplistic one-dimensional nature of the levels and believe the theory fails to describe the diversity evident in students' behaviour (Pegg, 1995). Despite these forms of criticism, there is considerable empirical support for the levels (Clements & Battista, 1992).

The van Hiele Theory hypothesises five levels of thinking identified as Levels 1 to 5. Of interest to this study are the first three levels. These are described below.

Level 1: Figures are identified according to their overall appearance.

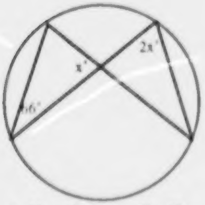
Level 2: Figures are identified in terms of properties which are seen as independent of one another.

Level 3: Relationships between previously identified properties of a figure are established as well as relationships between the figures themselves.

One issue that has confronted the writers is that the current level descriptors are narrow and not easily generalisable to a range of question types common in school geometry. When one considers some of the typical questions asked of students in the junior secondary school there does not appear to be much guidance from the van Hiele Theory in allocating levels. Reference to van Hiele's writings or the two most commonly used tests, i.e., those devised by Usiskin (1982) and Mayberry

(1981) do not help. For example, the following four multiple choice questions, taken from a State examination in New South Wales, typify the concern.

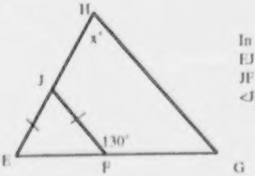
Question 1



x = ?

A. 33 B. 38 C. 48 D. 57

Question 2

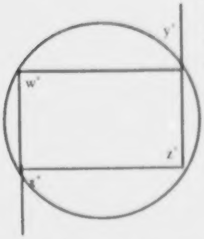


In the diagram
EJ = JF
JF ∥ HG
<JFG = 130°

x = ?

A. 50 B. 80 C. 100 D. 130

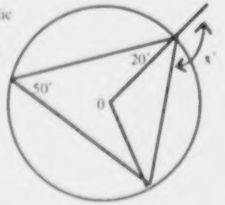
Question 3



The diagram shows a cyclic quadrilateral.
Which statement below is always true?

A. z = w
B. x = w
C. x + y = 180
D. x + w = 180

Question 4



The diagram shows a circle centre 0
x = ?

A. 110
B. 120
C. 130
D. 140

These items were given, as part of general mathematics examinations to some 35,000 students (who represent 40% of the Year 10 (16 year old) student population) doing the advanced mathematics course in the state of New South Wales, Australia. Significantly, a consistent pattern emerged in the nature of responses. This showed (i) students who were correct on Questions 2 (approx 75% of the candidature) were also able to handle correctly and consistently typical Level 2 items, and (ii) students who were correct on Questions 1 and 3 (approx 45% of the candidature) performed correctly and consistently on more typical Level 3 items. Approximately 60% of the candidates were correct with Question 4.

While there was clearly a close relationship between these items and van Hiele's Levels 2 and 3, van Hiele's descriptors give little guidance to support coding the nature of the thinking associated with these four questions.

One approach to addressing the issue of more inclusive level descriptions is to refer to a theory that is seen to have some sympathy with the van Hiele Theory. The SOLO model (Biggs & Collis, 1982, 1991) has been identified by several writers (e.g., Jurdak, 1989; Olive, 1991; Pegg & Davey, 1989) as having strong similarities with the van Hiele Theory, despite philosophical differences.

The SOLO Taxonomy, as with the van Hiele Theory, has its roots in the Piagetian tradition and both theories are relevant to, and designed to facilitate, school-learning activities, albeit in different ways. The van Hiele levels are a series of signposts of cognitive growth reached through a teaching/learning process as opposed to some biological maturation. SOLO, however, is particularly applicable to judging the quality of instructional dependent tasks. It is concerned with evaluating the quality of students' responses to various stimulus items. While it is possible to set questions which encourage a response at a particular level, it is students' attempts at an item that are of paramount interest as well as the many natural groupings of answer types. This focus represents an important departure from, say, Bloom's Taxonomy where levels have an *a priori* quality and students are deemed either successful or not. With SOLO, the data (students' responses) are treated polychotomously (as opposed to dichotomously, i.e., true or false) and the categorisation of answers into multiple groupings with similar characteristics reflects various stages of cognitive growth.

This represents a philosophic shift from van Hiele's (and Piaget's) ideas, as the levels describe responses, not people. With SOLO, a response provides a measure of a student's attainment at a particular time and in a particular circumstance, it does not, necessarily, determine some stable stage of cognitive functioning. This approach overcomes the décalage problem identified by Piaget in which a person may respond at a different level to the same (or similar) tasks from one testing episode to the next.

Fortunately, this difference in focus between SOLO and the van Hiele Theory does not represent a fatal flaw to any comparison of level descriptions across the two frameworks. The main difference between them is manifested in the conclusions that are drawn about the overall nature of a student's level of thinking. This contrasting view is not the focus of this paper and useful comparisons about levels can be made without confronting this issue.

A SOLO classification combines two aspects. The first of these is the mode of functioning and, the second, a level of quality of response within the targeted mode. Of relevance here is the mode referred to as concrete symbolic and the three levels within the mode referred to as unistructural, multistructural and relational.

In the concrete symbolic mode a student is capable of using, or learning to use, a symbol system, such as a written language and number notation. The important feature of this mode is that there is an empirical referent available. This is the most common mode addressed in learning in the upper primary and secondary school. This mode becomes available after an individual has progressed to a certain level of attainment within the ikonic mode. (The ikonic mode is where a person internalises outcomes in the form of images. It is in the ikonic mode that a child develops words and images that can stand for objects and events.) Hence, early responses in the concrete symbolic mode carry with them the need to base judgments on observable, physical experiences that make sense to the real-world understanding of the person.

The three levels within the concrete symbolic (C.S.) mode (Biggs & Collis, 1991), represent a growth from the more 'concrete' to the more 'abstract'. Brief descriptions of these levels are:

The *unistructural level* of response is one that contains, or draws upon, one relevant concept or datum from among all that was available.

The *multistructural level* of response is one that contains several relevant but isolated concepts or data from among all that was available.

The *relational level* of response is one that contains an over-riding linking concept. Alternatively, each relevant concept is woven together to form a coherent structure.

The development encompassed by the three SOLO levels moves from a focus on a simple (one aspect) and more tangible aspect, which is closely aligned with an individual's real-world experience, to the less tangible aspects, namely, relationships between concepts. This growth is, in part, determined by the availability of working memory for the completion of the task. At the unistructural level the student has only to understand the question, relate the question and the answer, and use one concept. The multistructural level response requires a similar ability except the student needs to be able to access a number of concepts. The relational response requires, in addition, an overview of relevant concepts while being able to monitor the process or task from beginning to end, thus allowing for a logically complete conclusion.

Finally, notions of consistency and closure are also strongly related to each level. The former refers to the need felt by individuals to make a response that is not contradictory, either from the perspective of the answer or the data provided. The latter refers to the desire to provide an answer and hence finish the task. These two needs represent opposing forces that impact on the nature or quality of the response.

The link between the level descriptors of SOLO and the van Hiele Theory have been summarised recently elsewhere (Pegg and Davey, in press) and in summary it is: unistructural responses (C.S. mode) and multistructural responses (C.S. mode) are associated with Level 2 thinking; and, relational responses (C.S. mode) with Level 3 thinking.

This link has identified a way forward to expand upon the descriptions offered by van Hiele for his Levels 2 and 3. In essence, this development means, in the case of Level 2, that the characteristic of thinking in terms of independent properties can be interpreted within SOLO as an aspect of a broader thinking category in which concepts are addressed in isolation. These concepts need to have an obvious visual basis and individual closures (answers) must have a strong real-world referent for students. Aspects of processing which occur towards the end of a task appear independent of any initial processing. The individual steps (or closures) leading to a final solution can be performed in sequence without concern given to some general overview. A distracter in the process can cause concerns.

At Level 3, thinking identified by van Hiele concerns the acceptance and use of relationships between properties and figures. The broadening of this level using SOLO is associated with the ability to have an overview of relevant elements and to form, on this basis, appropriate generalisations. Consequently, for a given task, relevant data are identified and the student can monitor these data. This allows for the reasoning at a later stage in a question to be adjusted in the light of earlier thinking. Also at this level, students have a notion of a generalised number by using algebraic symbols which can stand for 'real' numbers. Hence, they do not, necessarily, replace pronumerals with numbers but students feel secure as they have the option to do this in cases in which they perceive to be more difficult. This ability to work with pronumerals allows students to refrain from the need to calculate particular answers for each step of a problem (a characteristic of Level 2 thinking), and opens the way for relationships between different concepts to be utilised.

Interestingly, support for these latter ideas can be found in van Hiele's own writings although he chose not to pursue this aspect. He stated "The differences between the objects of the second and third levels can also be demonstrated by different ways of writing. At the second level, calculation deals with relations between concrete numbers: $4 \times 3 = 12$, $6 + 8 = 14$. At the third level of thinking it deals with generalisation of results: $a \times (b + c) = (a \times b) + (a \times c)$. In these generalisations you do not return to the original objects of the second level, namely the concrete numbers" (1986, p.54). Here, we can see van Hiele broadening his own level descriptions. For example, at the second level there is a focus on actual numbers, single concepts are involved, and working memory demand is relatively light. At the third level, the working memory demands are heavier, the example given is more abstract in nature and actual numbers are not used. The rule cannot be known by its separate parts, only by an overview of all the elements and the structure of the relationships can the pattern be understood. The letters in the equation each represent an entity in their own right; there is no direct concrete referent for the rule although substitution of numbers remains an option which would guarantee uniqueness of outcome.

The remainder of this paper reports on the results of a longitudinal study which, in part, explored within an interview situation the responses of 12 fifteen year olds, who were selected to represent the bottom, middle and top 20% of the age cohort, to the four questions stated previously. A student has been selected from each group and their responses to the four questions, given on two occasions some six months apart, are reported and discussed.

Results

Sam In his first interview session Sam relied entirely on the look of the angles in the diagram for each of the four questions. For example, in Question 1 he stated, "Being that that angle there is 66° and that looks a little bit smaller than that one and bigger than 38 so I think 57 is the appropriate (one)." This answer represents a transition in thinking between van Hiele's Levels 1 and 2 (and could be coded between the ikonic and concrete symbolic mode using SOLO) as the student is not solely relying on visual skills but is aware of angle sizes which he can interpret in terms of degrees.

In the second interview session Sam showed little change. There was an indication with Question 1 that he was trying to move beyond visualisation techniques and he focused on the two data elements of $2x^\circ$ and 66°. He stated, "um it would be equal to 33 because you have $2x$ and 66 degrees, they would have like x and you have like two of them, then that is 33 to equal 66." Such thinking would be seen as very early Level 2. For all the other questions he relied on the required angle's visual appearance as he had done in the first interview.

Ann In her first interview session Ann was able to complete successfully Question 2 but was unsuccessful in the other three questions. In Question 2 she successfully undertook a series of tasks, such as finding the base angles of the isosceles triangle, then the third angle, and, finally, she used the notion of equal corresponding angles to find x.

Question 1 was seen to be more difficult. She attempted to use 'opposite angles' but on realising this was not correct she was not able to continue. "These two angles are opposite so they have to be the same. Um this side equals that side, oh no it doesn't (pause). I can't do that one." For Question 3 she chose B citing corresponding angles and assuming parallel lines. In Question 4 she linked a number of facts, such as one angle being 160°, two radii being equal, and the angles opposite these being equal, but she was unable to sequence the ideas.

Overall, there was a consistency in her approach to the questions. A form of Level 2 thinking was demonstrated in all questions. For Question 2 the task was straightforward enough for her to sequence the steps. Her lack of familiarity with circle properties precluded this happening in Question 4, and for Questions 1 and 3 the cognitive load was such that she could only undertake a single process which, in each case, was inappropriate.

Ann's performance improved marginally in her second interview. She was correct on both Questions 2 and 4. In this latter question she started with the 50° angle and ignored the 20° angle. "Um if x is 50 this must be 100 because that is twice that, and these two lines are equal because they are both radius of a circle so that is an isosceles triangle so both these angles must be 40 so um angles on a straight line are 180 so that must be 140."

While the structure of her response to Question 1 remained the same (i.e., a focus on one issue) the feature chosen was more clearly related to the question than in her earlier attempt.

Ann: *I think that these two angles are the same, um but if it is these two angles the same than x has to be 33°.*

Int: *What makes you think that $2x$ and 66 are equal?*

Ann: *Um when we did circle geometry I remember something about these angles that are way off in the corner being equal, but I don't know if it these two or these two, I can't remember.*

For Question 3 she offered D as the solution but could not provide a reason.

Overall, Ann's second interview showed her consolidating her ability to solve Questions 2 and 4, and she was more directed in her attempts to solve the other questions. This means that when the cognitive load was lighter she could undertake a series of tasks and order these tasks - Level 2 thinking. As the cognitive load increased she was able to focus only on a single relevant issue, i.e., early Level 2.

Chris In the first interview Chris was correct for both Questions 1 and 2 and he made a reasonable attempt at Question 4 but could not attempt Question 3. For Question 1 he found the missing two angles in the triangle containing $2x^\circ$. First, he used the 66° and then the $2x^\circ$. Seeing a triangle with all angles marked prompted him to use the angle sum of a triangle from which he found the correct answer.

The assumption in Question 4, that the large triangle was isosceles, caused an error. This lead to Chris using the 20° angle to determine the base angles of the small isosceles triangle to be 45°. As a result x ° was found to be 135°.

Chris demonstrated relatively sound Level 2 skills in the three questions he attempted. In each case he was prepared to move forward from the known angles until he reached a value for the unknown. He always commenced the question with a known specific angle. While this strategy is generally successful it caused him a problem in Question 4 when he chose the wrong angle to start and in Question 3 where there was no specific angles available to choose.

In the second interview, all questions were correct although he was unable to explain his attempt for Question 3. Of interest was the fact that in Question 1 he chose to identify the angle equal to $2x$° first and then complete the angle sum. In doing this it was clear he had an overview of the question and he knew that the answer would proceed from the process he put into place. This was very different to the ad hoc approach he adopted on this question in the first interview. Also of interest was his choice to begin Question 4 with the 50° angle (and to ignore the 20° angle) and then move efficiently to obtain the correct response.

Overall, Chris was now showing signs of early Level 3 thinking. He was able to efficiently undertake sequential tasks and was showing signs of being able to monitor a number of relevant elements in questions. Problems occurred when the data seemed to be too removed from real-world referents, as was the case with Question 3.

Conclusion

The four questions posed earlier in the paper can now be discussed within van Hiele's framework. In doing this, the difficulties and successes students experienced with these questions can be interpreted.

The two questions, Questions 2 and 4 can be solved by students at Level 2. To complete successfully these tasks, the student must be aware of the relevant concepts and be able to employ more than one concept. There is a clear, unique end-point, the use of actual angle measurements ensures a series of 'real', meaningful closures for the students. The questions can be solved without the need of an overview of the questions and the available data. Students can be successful by moving logically along a path from the given angle to the answer without needing to know in advance how they might proceed. Indeed the path to the answer would be unexpected until the completion of the activity. The reason for the increased difficulty for Question 4 is related to either unfamiliarity with circle properties or by the distraction caused by the 20° angle. This latter error sets up some interference to the solution process with students attempting to start with this angle or, alternatively, using it part way. Absence of an overview means such information can set a student on an unsuccessful solution path from which they cannot recover.

It is possible to complete Question 1 using Level 2 thinking (see Chris's first interview). However, there is a trial and error feel about the approach and the student's success rate is very inconsistent when undertaking similar questions. For a student to be routinely correct on Questions 1 and 3, thinking at Level 3 is required. The major advance in thinking appears to be the ability to refrain from seeking individual closures (answers) before proceeding to the next step. This represents an ability to form a generalisation based on previous 'concrete' experiences with specific cases although final uniqueness of the result still needs to be guaranteed. This level also represents an ability to work with pronumerals. In each case this usage can be supported by replacement of some unique numbers, which make sense to the student within a question's context, if the student feels pressured. In both questions, students who are consistently successful at this type of problem have an overview of all the elements of the question. They are aware of the correct solution strategy which relates the use of various concepts before they start the question and, unlike those who perform at Level 2, the arrival at the final answer would not come as a surprise.

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CLASS DISCUSSION AS AN OPPORTUNITY FOR PROPORTIONAL REASONING

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Abstract

The initial phase of the construction of proportional reasoning with students aged 12 to 13 is described. In the proposed problematic situation, the recourse to the constance of ratios originates as a strategy necessary to tackle this same situation and more importantly it clashes with other resolutive strategies which are quite spontaneous but not really suitable.

The fundamental part of the didatic proposal turned out to be the discussion conducted by the teacher, which gave the students the opportunity to freely express their agreement or disagreement, even with original argumentation from the mathematical point of view.

In reference to current constructivist perspectives, the main points of contact are underlined, with a particular emphasis on social constructivism.

1. Introduction

At primary school, the students learn to recognise the concept of ratio with the so-called "partitive" or "quotitive" meanings (Nesher, 1988, Silver, 1986). This picture is gradually extended to the junior high school, where in the area of proportionality between sizes (homogeneous or not), the concept of ratio can be applied to many other situations in mathematics, physics and other areas.

The fact is that text books tackle this theme in a way which can be defined as dogmatic, because the theory presented is put forward in most cases without any kind of argumentation or justification (Triulzi, 1995). Also, the text books usually make the student go through a large variety of exercises which they learn to perform in a mechanical way: the same arrangement of exercises, in the section dedicated to ratio and proportionality, certainly doesn't challenge the student to utilise other resolutive strategies instead of ratios.

The concept of ratio and in general of proportionality, covers a particularly important area in mathematics, because it undoubtedly constitutes a central concept which has many applications even in high schools and universities.

It is only during the course of the junior high school however, that the students have the opportunity to study ratio and proportionality between sizes as a separate subject and to tackle also, among various types of problems, situations which call for a proportional reasoning: therefore it is necessary to dedicate time to it and to pay particular attention to its conceptualisation.

At high school, even when the subject of proportionality between sizes is taken up again, it is assumed that the students know the problems and it is taken for granted that they know how to identify without hesitation the situations which call for the use of such a concept.

So it is right then, during the course of junior high school that a didactic intervention on this subject can be inserted, aimed at challenging, before even presenting, a way of reasoning which relies on the constancy of ratios.

The didactic proposal which we have devised for the second year of junior high school sees the extension of the concept of ratio and the construction of proportional reasoning; a construction which has been worked out, discussed and agreed upon by the whole class, through the exploration of appropriate problematic situations.

In this paper, the first part of this didactic proposal will be described, after a three year experimentation period with students aged 12 to 13 (the complete experience is described in Castagnola, 1995 and Torresani, 1997). In particular, the first worksheet for the students will be described and displayed, as well as the main results gathered from the students' protocol and the ideas which emerged in class during discussions led by the teacher.

2. Theoretical Framework

In reference to the abundance of literature available on proportional reasoning and on the didactics relating to this subject (see References), the itinerary planned by our research group focuses particularly on the initial phase of exploration of specific problematic situations and especially on the discussion of the resolutive strategies which emerged in class. The role of the discussion, conducted properly by the teacher, is considered fundamental and decisive to the comprehension of mathematic concepts, in our case to the conscious construction of proportional reasoning.

In reference to French literature, the discussion was conducted following the so called a-didactic modality (Brousseau, 1986): it means that during this phase of interaction between students the teacher does not take any position with respect to the knowledge involved. In addition the teacher plays different roles, according to different didactical phases: he (she) coordinates the discussion, solicits pupils for explanation, stresses different positions, promotes peers' verbal interaction (Arzarello et al., 1996, Bartolini Bussi et al., 1995).

Therefore the dialogue and the negotiation of meanings during mathematics class have a central role, according to the principles of social constructivism sustained by Bauersfeld (1995), Cobb, Yackel, Wood (1992) and Ernest (1995).

The experimentation was carried out also in accordance with those researchers who underline the role of pupils' errors as a source of investigation. With particular reference to Borasi (1996) we did indeed consider students' unappropriate solution strategies as the source of opportunity for mathematical investigation, not as something to be hidden or erased: the atmosphere of inquiry was established and maintained in class thanks only to these incorrect strategies.

The basic ideas discussed and shared by the teachers involved in the experience, who took part from the beginning in the realization of the study, are summarized in the following issues:

- the teacher is the person who helps students in the process of personal construction of mathematical ideas;
- the teaching process is active and constructive, during which students face problems posed by the teacher but that they themselves decide to solve;
- the teaching - learning process is interactive, characterized by continuous negotiation of mathematical meanings: the interaction takes its origin from the discrepancies between students' solution strategies;
- the language and its interpretations negotiated in class are therefore like mediators, during the knowledge construction that students perform through dialogue, between their cognitive processes and the experience they live in the class.

During the conducted experience the class discussion, following the individual work on worksheets, had therefore the main role, as stressed before. It may be important, in relation to this phase, to clarify more precisely the role of the teacher, which is that of coordinator. The teacher

- encourages each student to motivate his (her) answer or solution;
- calls upon the students to speak when they require to do so;
- invites students to listen and to express themselves on the positions of schoolmates;
- promotes the negotiation of the meanings related to the used words;
- syntesizes (eventually) different positions emerged in the class;
- does not take any position on the correctness of students' proposals but solicits everyone to reflect on each of these proposals.

Only at the end of the process (which could require more class discussion) the teacher formulates, together with the class, the concept or the procedure which is the object of discussion. This happens, in any case, in a way that takes into account both the usual mathematics shared by society and expecially the activity worked out together.

In what follows the pupils' main interventions during the first discussion are reported, with the aim of higlighting the nature and sometimes the originality of their argumentations from the point of view of mathematics connected with the situation they faced. It is interesting to reflect upon the maturity of the argumentations that students spontaneously put forward, when the teacher gives them enough time to reflect and to discuss. We believe this is the result of the positive and constructive atmosphere, established in class by the teacher, consequent to the theoretical principles mentioned before.

The different roles of the teacher, which could be described following her interventions throughout the discussion, will not be described here: this should require the complete reproduction of the discussion protocol.

3. Students' intervention during the first part of the didactic itinerary

For the first worksheet, which was aimed at testing spontaneous recourse to the use of the ratio concept, we decided to use a context which was quite familiar to the students, not uncommon in school tests and frequent in literature: the mixing of colours. The choice of the context and its verbal formulation required long discussion within the research group: it had to be simple from the arithmetic point of view and it had to promote more solution strategies. In addition, the meaning of ratio related to the situation had to be different from "partitive" or "quotitive" meanings. The following is the text which appeared on the worksheet:

Three panels of different dimensions have to be painted and equal size tins of yellow and blue are available. The panels have to be painted the same shade.

MARCO painted the first panel using a colour obtained by mixing 4 tins of blue and 6 tins of yellow.

LUISA has to paint the second panel: to obtain the same shade of the colour and with 6 tins of blue available, how many tins of yellow does she need?

PIERO has 3 tins of yellow for the third panel. How many tins of blue does he need?

Explain your reasoning in answering the questions:

For LUISA ....

....

....

For PIERO ....

....

....

It may be that students face the proposed problem using intuitive strategies which prelude the proportional reasoning: some could appeal explicitely to the constancy of ratios between the given quantities. In any case, from the answers to this worksheet the teacher derives information about the different cognitive levels in the class group. This is therefore an introductive worksheet, whose aim is that of promoting a profitable discussion.

The following results are usually obtained from our experimentation classes:

- the majority of the students apply the criteria of the "constant difference", that is, they maintain in every case that the number of yellow tins is equal to the number of blue tins plus 2. Therefore, they say that Luisa needs 8 tins of yellow while Piero needs 1 tin of blue;
- some students keep the total number of tins (10) to be used constant ("constant total" stategy): Luisa needs 4 tins of yellow and Piero needs 7 tins of blue.
- only a minority of students make recourse to intuitive strategies which deal with proportional reasoning. Some observe, for instance, that Luisa's blue tins (6) are one and a half times the amount of Marco's (4), therefore it has to be the same for the yellow tins: if Marco has 6 tins, Luisa has to have 9. In the case of Piero the situation is simpler: he has 3 yellow tins, half the number of Marco's, so he also has to have half the number of blue tins, that is 2, half of 4.

Right at the end of the individual work, the teacher invites pupils to read their solution strategies, to explain them and to discuss discrepancies.

What follows are the most significant passages of the discussion carried out in one of the classes involved in the experimentation.

At the beginning of the discussion, the different types of solution strategies described above are put forth by students and then the teacher asks them if all three strategies could be correct or not.

Valeria: *In my opinion they cannot all be correct.For instance Simone's strategy* (who used the 'constant total' strategy) *is not right because if Piero has 3 tins of yellow and 7 of blue and mixes them together, the result is not green, because the blue is darker and so the result is blue.*

Teacher: *So, in your opinion, where has Simone gone wrong?*

Valeria: *He has mixed more blue than yellow.*

Alberto: *I think that Jacopo* (who used the'constant difference' strategy) *is wrong because here it is said that the panels have to be of the same shade of green and in order to have the same shade, it is necessary to have the same ratio of blue tins to yellow tins.*

Teacher: *Jacopo, in your opinion, is Alberto right or not?*

Jacopo: *I don't know.*

Giovanni: *I don't think he is right, because it doesn't matter how many tins there are. If the difference is always 2, the colours are always the same, even if the quantity changes.*

Jacopo: *There are not always the same number of blue tins, but if you always add +2 of yellow, the shade, in my opinion, comes out the same.*

Alessio: *I want to say something interesting. Imagine you have to paint a house in the same shade of green. If you use 1000 buckets of blue and 1002 of yellow, do you think you'll get the same shade of green ?*

Giovanni: *Yes, I do.*

Alessio: *Really ?*

Giovanni: *Yes, the quantity will change but the same green will come out.*

Pamela: *In my opinion what Jacopo says is wrong but I don't completely agree with Alberto, because here it is said that the panels don't have the same dimension, so if you use less tins, both of blue and of yellow, but always in the right quantity, then there is never a different green.*

Paolo: *But in fact he does not always use the same tins. For instance if you have a panel twice the size, you will also use twice the amount of paint: that is, 4x2 tins of blue and 6x2 tins of yellow.*

Niccolò: *Thanks to Alessio's example of the house, I have understood that to get the same shade you have to maintain the same ratio, because I am not convinced that by adding 1002 tins of yellow to 1000 tins of blue you'd get the same green.*

Alessio's contribution is of a theoretical nature: he wants to demonstrate that when dealing with a large quantity of paint where the difference between blue and yellow is always two, the quantity of each colour is almost the same, so it is impossible for the final colour to be the same as that obtained from 4 tins of blue and 6 of yellow. It is interesting to note that Alessio has used, in his protocol, the 'constant difference' strategy, but at the beginning of the discussion he realized its incorrectness and became a convinced promoter of the 'constant ratio' strategy.

In class there is confusion: Alessio's example has created two opposed factions: one in favour and one in opposition to Alessio. Jacopo wishes to speak.

Jacopo: *Using 1000 litres of blue and 1002 litres of yellow or mixing 1 litre of blue and 3 litres of yellow, I believe, you obtain the same shade.*

Teacher: *Do you agree with Jacopo ?*

A chorus of 'yes' and 'no' breaks out. No one backs down from his or her position. The problem is how to decide if the same shade of colour, in the proposed situations, is maintained or not. Paolo requires to speak.

Paolo: *If you have 2 tins of blue and 2 tins of yellow you get one shade of green. If you add another 2 tins of yellow you get a lighter green, almost yellow. But if you have a lot of green and you add 2 tins of yellow, you don't get such a yellowish green, but the original green colour remains almost the same.*

What Paolo means is that 2 tins of yellow have much less of an effect on a lot of green than on the mixture of 2 tins of blue and 2 tins of yellow. His intervention is in accordance with Alessio's argumentation: they both stress that the effect of 2 tins of yellow on a little quantity and on a big quantity of the same mixture cannot be the same.

At this point, the discussion is interrupted as the lesson has come to an end. Some schoolmates appreciated the interventions of Alessio and Paolo and have expressed their agreement (4 students, during the discussion, have passed from the strategy of the 'constant difference' to that of the 'constant ratio') but it is not exactly the same for the whole class group: the construction of proportional reasoning is only at the beginning.

Summarizing the results of the discussion, the strategy of the 'constant total' has been rejected also by the respective promoters and only the strategy of the 'constant difference' and that of the 'constant ratio' remain: it is not yet clear which of these is right, even if interesting argumentations against the use of the difference have emerged. The class is ready to face proportional reasoning in the different contexts of successive worksheets.

It is not possible to give details here of how the itinerary was developed, but it seems of some interest to report another of Paolo's interventions during the successive discussion, centered on another proposed situation. With the aim of contrasting the strategy of the 'constant difference' supported by some schoolmates, Paolo refers again to the first situation of colours and suggests the following type of reflection:

Paolo: *I would say to Giovanni* (who is convinced that the strategy of the 'constant difference' is right)*: you say that, in order to have the same shade, the yellows have to be always 2 more than the blues. But if there are 2 yellow tins, there have to be 2 less blues, that is, 0. It is not possible to obtain the same shade: yellow comes out. Therefore the same difference does not work.*

This time Paolo makes recourse to a different and interesting argumentation to contrast the 'constant difference' strategy and his explanation is very clear.

In conclusion, it seems important to stress that the students' argumentations are all of a theoretical nature: they don't have real colours to try, but they have the skill to imagine the situations, to construct mathematical models, to make analogies, to force the models to their limits and to interpret them in the given situation, confuting some strategies with an original recourse to extreme cases (using 0 and 1000).

On the basis of successive class work, it would be possible to conclude that especially the recourse to 0 must have had a noticeable effect on the pupils because often similar motivations to Paolo's emerged with the aim of contrasting the 'constant difference' strategy.

3. Concluding remarks

It seems important to underline some features which characterize the activity proposed to pupils and allow us to define it as a 'problematic situation' (in the sense specified by Jaquet, 1993):

- The initial knowledge of the students was judged to be sufficient in order for them to proceed by themselves: that is, to use a strategy which refers (even unconciously) to proportional reasoning or another type of 'pattern', which in any case they were able to accept or to reject.
- The pupils were able to decide for themselves (to 'validate') if a particular solution was correct or not: the discussion was in this case fundamental, starting from the strategies proposed by the students themselves.
- The pupils had to build a new concept: the aim was to reach proportional reasoning.

- The knowledge which we wanted the students to acquire, was the most suitable means of solving the problems proposed: the additive strategies were inadequate and the most suitable mathematical method was recourse to constant ratio.

Also in accordance with teachers' judgment, the mentioned features made the cognitive participation of the class and the success of the experience possible.

In reference to the way of carrying out the didactic program through class discussions, the following positive aspects were specified by the teachers:

- there was a major participation of the class in the proposed activity;
- students were aware of their direct involvement in the construction of their knowledge;
- leaving a problematic situation open (sometimes for a long time), enabled every student to have adequate time for reflection and many opportunities for expression and comparison;
- leaving the students to speak freely put the path of their learning processes in evidence and also the cognitive obstacles they met;
- the interaction between peers developed their skill of reasoning in formulating their own point of view and of interpreting that of schoolmates (educating them to respect and compare others' ideas);
- there was an awareness, on behalf of the teacher and of the students, that mathematics was being constructed through activities which were exclusive to their class.

Concluding the analysis of the features related to the described didactic procedures, it seems important also to mention two unresolved problems. First of all the didactic material for realizing such a project has to be prepared entirely: the usual textbooks, in their nature, cannot propose situations with a development similar to that of a particular class. In addition (and this is also a wider problem) teachers are not yet prepared to realize such an educational project: the acceptance and the preparation for a new teacher's role and a new way of working in class are necessary. Some efforts have been made in this direction, but a lot of work has yet to be accomplished.

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